Harmonic Functions via Linear Algebra

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reference

Sheldon Axler and Peter Shin, The Neumann Problem on Ellipsoids, *Journal of Applied Mathematics and Computing* 57 (2018), 261–278.

Peter Shin's institutional affiliation at the time was

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Notation: $w = (x, y, z) \in \mathbf{R}^3$.
Suppose E is an open subset of $\mathbf{R}^3$.

Definition: *Laplacian*

For $u\colon E \to \mathbf{R}$, the *Laplacian* of u is denoted Δu and is the function $\Delta u\colon E \to \mathbf{R}$ defined by

$$(\Delta u)(w) = \frac{\partial^2 u}{\partial x^2}(w) + \frac{\partial^2 u}{\partial y^2}(w) + \frac{\partial^2 u}{\partial z^2}(w).$$

u is called *harmonic* on E if

$$(\Delta u)(w) = 0$$

for all $w \in E$.

examples of harmonic functions

u is harmonic if

$$\frac{\partial^2 u}{\partial x^2}(w) + \frac{\partial^2 u}{\partial y^2}(w) + \frac{\partial^2 u}{\partial z^2}(w) \equiv 0.$$

Example 1: The function $u\colon \mathbf{R}^3 \to \mathbf{R}$ defined by

$$u(x,y,z) = 6x^2yz - y^3z - yz^3$$

is harmonic on $\mathbf{R}^3$.

Example 2 (Newton): The function

$$(x,y,z) \mapsto \frac{1}{\sqrt{x^2+y^2+z^2}}$$

is harmonic on $\mathbf{R}^3 \setminus \{0\}$.

Example 3: The function $u\colon \mathbf{R}^3 \to \mathbf{R}$ defined by $u(x,y,z) =$

$$23x^3y^2z - 8xy^4z - 7xy^2z^3 - 2x^5z - x^3z^3 + xz^5$$

is harmonic on $\mathbf{R}^3$.

Example 4 (Poisson): If $\zeta \in \mathbf{R}^3$ and $\|\zeta\| = 1$, then

$$w \mapsto \frac{1 - \|w\|^2}{\|w - \zeta\|^3}$$

is harmonic on $\mathbf{R}^3 \setminus \{\zeta\}$.

Neumann problem

Neumann problem: Given $f: \partial E \to \mathbf{R}$ continuous, find continuous $u: \overline{E} \to \mathbf{R}$ such that u is harmonic on E and the normal derivative of u on ∂E equals f.

Fix positive numbers a, b, c. Let E be the ellipsoid

$$E = \{(x, y, z) \in \mathbf{R}^3 : ax^2 + by^2 + cz^2 < 1\}.$$

Let

$$q(x, y, z) = ax^2 + by^2 + cz^2.$$

The outward pointing normal at a point (x, y, z) on ∂E is

$$(\nabla q)(x, y, z) = (2ax, 2by, 2bz).$$

Neumann problem: Given $f: \partial E \to \mathbf{R}$ continuous, find continuous $u: \overline{E} \to \mathbf{R}$ such that u is harmonic on E and

$$(\nabla u)(w) \cdot (\nabla q)(w) = f(w)$$

for every $w \in \partial E$.

Neumann problem on ellipsoids

Let A denote surface area measure on ∂E.

Let $\mathcal{P}_m$ denote the vector space of polynomials on $\mathbf{R}^3$ of degree at most m.

Let $\mathcal{H}_m$ denote the vector space of harmonic polynomials on $\mathbf{R}^3$ of degree at most m.

Proof (b) $\Longrightarrow$ (c) is obvious.

For (c) $\Longrightarrow$ (a), use Green's Second Identity.

To start the proof of (a) $\Longrightarrow$ (b), recall that

$$(\nabla q)(x, y, z) = (2ax, 2by, 2bz).$$

Neumann problem solution

Suppose $f \in \mathcal{P}_m$. Then the following are equivalent:

(a) $\displaystyle\int_{\partial E} \frac{f}{\|\nabla q\|}\, dA = 0.$

(b) There exists $u \in \mathcal{H}_m$ such that

$$\nabla u \cdot \nabla q = f \text{ on } \partial E.$$

(c) There exists a harmonic function u on $\overline{E}$ such that

$$\nabla u \cdot \nabla q = f \text{ on } \partial E.$$

proof that (a) implies (b)

Neumann problem solution

Suppose $f \in \mathcal{P}_m$. Then the following are equivalent:

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(b) There exists $u \in \mathcal{H}_m$ such that

$$\nabla u \cdot \nabla q = f \text{ on } \partial E.$$

(c) There exists a harmonic function u on $\overline{E}$ such that

$$\nabla u \cdot \nabla q = f \text{ on } \partial E.$$

Define a linear map $T\colon \mathcal{H}_m \to \mathcal{P}_m|_{\partial E}$ by

$$Tu = (\nabla u) \cdot (\nabla q)|_{\partial E}.$$

$\dim \operatorname{null} T = \dim$ {constant functions} = 1.
Thus

$$\dim \operatorname{range} T = \dim \mathcal{H}_m - 1 = \dim \mathcal{P}_m|_{\partial E} - 1.$$

Define a linear functional $\varphi\colon \mathcal{P}_m|_{\partial E} \to \mathbf{R}$ by

$$\varphi(f) = \int_{\partial E} \frac{f}{\|\nabla q\|}\, dA.$$

Because (b) implies (a), we have

$$\operatorname{range} T \subset \operatorname{null} \varphi.$$

But $\dim \operatorname{range} T = \dim \operatorname{null} \varphi$, and thus

$$\operatorname{range} T = \operatorname{null} \varphi.$$

Computing with Harmonic Functions

Sheldon Axler

In[1]:=
```
<< "C:\\Dropbox\\math\\publications\\HFT-Mathematica\\HFT12.m"
```

HFT12.m, version 12.03, 20 December 2020; for use with *Mathematica* versions 7 through 12 (and probably later versions of *Mathematica*).

The HFT12.m *Mathematica* package is designed for computing with harmonic functions.

Documentation for the use of this package and information about the algorithms used in it is available in the document titled *Computing with Harmonic Functions*, which is available in both nb and pdf formats.

The most recent version of this HFT12.m package and its documentation *Computing with Harmonic Functions* are available at http://www.axler.net.

For addional information about harmonic functions, see the book *Harmonic Function Theory* (second edition), by Sheldon Axler, Paul Bourdon, and Wade Ramey, published by Springer.

Comments, suggestions, and bug reports should be sent to axler@sfsu.edu.

* You can now use the functions in this package.

Neumann problems

Suppose $q(x, y, z) = 5x^2 + 3y^2 + 2z^2$. Find a harmonic function whose gradient dotted with the gradient of q on the ellipsoid $\{(x, y, z) \in \mathbf{R}^3 : q(x, y, z) = 1\}$ equals $x^3 y z^2$.

In[2]:=
```
neumann[x^3 y z^2, {x, y, z}, region -> quadratic[{5, 3, 2}]]
```

Out[2]=

$$\frac{1}{144\,767\,520}\left(36\,900\,x\,y - 20\,470\,x^3\,y - 197\,775\,x^5\,y - 103\,410\,x\,y^3 - 21\,330\,x^3\,y^3 + 84\,321\,x\,y^5 + 371\,640\,x\,y\,z^2 + 2\,041\,740\,x^3\,y\,z^2 - 779\,220\,x\,y^3\,z^2 - 631\,260\,x\,y\,z^4\right)$$

Notice the huge integers appearing in the solution of a problem whose input data contains only single-digit integers.

First we check that our alleged solution above is actually harmonic:

In[3]:=
```
Δ{x, y, z}[%]
```

Out[3]=
```
0
```

Now we want to verify that the gradient of our alleged solution dotted with the gradient of q agrees with $x^3\,y\,z^2$ on the ellipsoid $\{(x,\ y,\ z) \in \mathbf{R}^3 : 5\,x^2 + 3\,y^2 + 2\,z^2 = 1\}$:

In[4]:=
```
gradient[%%, {x, y, z}].{10 x, 6 y, 4 z}
```

Out[4]=

$$\frac{1}{402\,132}\left(1640\,x\,y - 2047\,x^3\,y - 30\,765\,x^5\,y - 8043\,x\,y^3 - 2844\,x^3\,y^3 + 9369\,x\,y^5 + 24\,776\,x\,y\,z^2 + 249\,546\,x^3\,y\,z^2 - 77\,922\,x\,y^3\,z^2 - 56\,112\,x\,y\,z^4\right)$$

The result above does not look much like $x^3\,y\,z^2$, so we subtract $x^3\,y\,z^2$ to see if we get a function that equals 0 on the ellipsoid $\{(x,\ y,\ z) \in \mathbf{R}^3 : 5\,x^2 + 3\,y^2 + 2\,z^2 = 1\}$:

In[5]:=
```
% - x^3 y z^2
```

Out[5]=

$$\frac{1}{402\,132}\left(1640\,x\,y - 2047\,x^3\,y - 30\,765\,x^5\,y - 8043\,x\,y^3 - 2844\,x^3\,y^3 + 9369\,x\,y^5 + 24\,776\,x\,y\,z^2 - 152\,586\,x^3\,y\,z^2 - 77\,922\,x\,y^3\,z^2 - 56\,112\,x\,y\,z^4\right)$$

The result above does not look much like 0, but it factors nicely:

In[6]:=
```
Factor[%]
```

Out[6]=

$$-\frac{x\,y\,\left(-1+5\,x^2+3\,y^2+2\,z^2\right)\,\left(1640+6153\,x^2-3123\,y^2+28\,056\,z^2\right)}{402\,132}$$

The expression above is clearly 0 on the ellipsoid $\{(x,\ y,\ z) \in \mathbf{R}^3 :\ 5\,x^2 + 3\,y^2 +\ 2\,z^2\ =\ 1\}$, showing that we indeed have the correct solution.

To illustrate a more complicated example, let $q(x) = 3\,x^2 + y^2 + 2\,z^2$. We start with the function $x^4\,y^2$, but this function does not satisfy the necessary integral condition of this ellipsoid for there to exist a solution for the corresponding Neumann problem. Thus we subtract an appropriate constant k, which we can find as follows:

In[7]:=
```
integrateEllipsoidArea[x^4 y^2, {3, 1, 2}, {x, y, z}]
```

Out[7]=

$$\frac{1}{315}\sqrt{\frac{2}{3}}\,\pi$$

In[8]:=
```
integrateEllipsoidArea[1, {3, 1, 2}, {x, y, z}]
```

Out[8]=

$$\sqrt{\frac{2}{3}}\,\pi$$

In[9]:=
```
neumann[x^4 y^2 - 1/315, {x, y, z}, region -> quadratic[{3, 1, 2}]]
```

Out[9]=

$$\frac{1}{2\,701\,782\,720}\left(1\,437\,395\,x^2-4\,145\,028\,x^4-2\,454\,945\,x^6+2\,056\,969\,y^2+30\,392\,244\,x^2\,y^2+33\,332\,535\,x^4\,y^2-4\,317\,208\,y^4-26\,323\,635\,x^2\,y^4+1\,477\,725\,y^6-3\,494\,364\,z^2-5\,522\,076\,x^2\,z^2+3\,491\,640\,x^4\,z^2-4\,488\,996\,y^2\,z^2-42\,053\,400\,x^2\,y^2\,z^2+4\,157\,760\,y^4\,z^2+1\,668\,512\,z^4+3\,517\,260\,x^2\,z^4+2\,851\,140\,y^2\,z^4-424\,560\,z^6\right)$$

In[10]:=
```
Δ{x, y, z}[%]
```

Out[10]=
```
0
```

Now we want to verify that the gradient of our alleged solution dotted with the gradient of q agrees with $x^4 y^2$ on the ellipsoid $\{(x, y, z) \in \mathbf{R}^3 : q(x, y, z) = 1\}$:

In[11]:=
```
gradient[%%, {x, y, z}].{6 x, 2 y, 4 z}
```

Out[11]=

$$\frac{1}{675445680}\left(4312185 x^2 - 24870168 x^4 - 22094505 x^6 + 2056969 y^2 + 121568976 x^2 y^2 + 233327745 x^4 y^2 - 8634416 y^4 - 131618175 x^2 y^4 + 4433175 y^6 - 6988728 z^2 - 27610380 x^2 z^2 + 27933120 x^4 z^2 - 13466988 y^2 z^2 - 252320400 x^2 y^2 z^2 + 16631040 y^4 z^2 + 6674048 z^4 + 24620820 x^2 z^4 + 14255700 y^2 z^4 - 2547360 z^6\right)$$

The result above does not look much like $x^4 y^2 - \frac{1}{315}$, so we subtract $x^4 y^2 - \frac{1}{315}$ to see if we get a function that equals 0 on the ellipsoid $\{(x, y, z) \in \mathbf{R}^3 : q(x, y, z) = 1\}$:

In[12]:=

$$\% - \left(x^4 y^2 - \frac{1}{315}\right)$$

Out[12]=

$$\frac{1}{675445680}\left(2144272 + 4312185 x^2 - 24870168 x^4 - 22094505 x^6 + 2056969 y^2 + 121568976 x^2 y^2 - 442117935 x^4 y^2 - 8634416 y^4 - 131618175 x^2 y^4 + 4433175 y^6 - 6988728 z^2 - 27610380 x^2 z^2 + 27933120 x^4 z^2 - 13466988 y^2 z^2 - 252320400 x^2 y^2 z^2 + 16631040 y^4 z^2 + 6674048 z^4 + 24620820 x^2 z^4 + 14255700 y^2 z^4 - 2547360 z^6\right)$$

The result above does not look much like 0, but it factors nicely:

In[13]:=
```
Factor[%]
```

Out[13]=

$$-\frac{1}{675\,445\,680}\left(-1+3\,x^2+y^2+2\,z^2\right)\left(2\,144\,272+10\,745\,001\,x^2+7\,364\,835\,x^4+4\,201\,241\,y^2+144\,917\,700\,x^2\,y^2-4\,433\,175\,y^4-2\,700\,184\,z^2-14\,220\,930\,x^2\,z^2-7\,764\,690\,y^2\,z^2+1\,273\,680\,z^4\right)$$