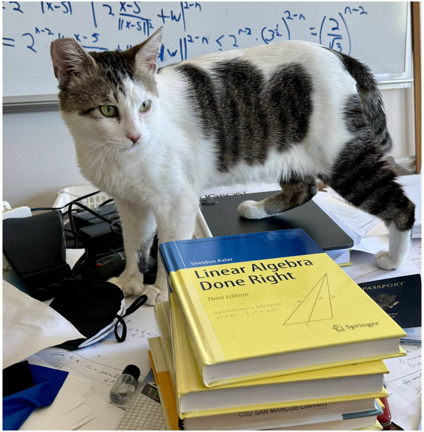
Unlocking Hidden Dimensions: The Power of Generalized Eigenvectors

Sheldon Axler

9 January 2025



Standing assumptions for this talk:

- V is a finite-dimensional complex vector space with
$$\dim V = n.$$
- $T\colon V \to V$ is a linear operator from V to V.

definition: *eigenvalues and eigenvectors*

- $\lambda \in \mathbf{C}$ is an *eigenvalue* of T if $T - \lambda I$ is not injective.
- A nonzero vector $u \in V$ is an *eigenvector* of T associated with λ if $(T - \lambda I)u = 0$.

definition: *generalized eigenvector*

Suppose $\lambda \in \mathbf{C}$ is an eigenvalue of T.

- A nonzero vector $u \in V$ is a *generalized eigenvector* of T associated with λ if
$$(T - \lambda I)^k u = 0$$
for some positive integer k.

Example: If $T\colon \mathbf{C}^3 \to \mathbf{C}^3$ is defined by $T(x, y, z) = (y, 0, z)$, then $T^2(x, y, z) = (0, 0, z)$. Hence $(x, y, 0)$ is a generalized eigenvector corresponding to the eigenvalue 0 if $|x|^2 + |y|^2 \neq 0$.

definition: ***generalized eigenvector***

Suppose $\lambda \in \mathbf{C}$ is an eigenvalue of T.

- A nonzero vector $u \in V$ is a *generalized eigenvector* of T associated with λ if
$$(T - \lambda I)^k u = 0$$
for some positive integer k.

If k is a positive integer and $T^k u = 0$ but $T^{k-1} u \neq 0$, then the list
$$u, Tu, T^2 u, \ldots, T^{k-1} u$$
is linearly independent.

Recall that $n = \dim V$.

only need to check n^{th} power

A nonzero $u \in V$ is a generalized eigenvector associated with λ if and only if
$$(T - \lambda I)^n u = 0.$$

Proof Apply comment at left with T replaced by $T - \lambda I$.

Then use the result that every linearly independent list in V has length at most n. ∎

It is not necessarily true that $\operatorname{null} T \oplus \operatorname{range} T = V$. However

$$\operatorname{null} T^n \oplus \operatorname{range} T^n = V.$$

V does not necessarily have a basis consisting of eigenvectors of T.

basis of generalized eigenvectors

There is a basis of V consisting of generalized eigenvectors of T.

Proof Induction on n. Assume true on all vector spaces of smaller dimension.

Let λ be an eigenvalue of T. Then

$$V = \operatorname{null}(T - \lambda I)^n \oplus \operatorname{range}(T - \lambda I)^n.$$

Choose a basis of $\operatorname{null}(T - \lambda I)^n$ (consisting of generalized eigenvectors of T).

Because $\dim \operatorname{range}(T - \lambda I)^n < n$, our induction hypothesis implies that there exists a basis of $\operatorname{range}(T - \lambda I)^n$ consisting of generalized eigenvectors of $T|_{\operatorname{range}(T-\lambda I)^n}$ (and hence of T).

Put these two bases together to get a basis of V consisting of generalized eigenvectors of T. ∎

definition: ***generalized eigenspace***

For $\lambda \in \mathbf{C}$, the *generalized eigenspace* $G(\lambda, T)$ corresponding to λ is defined by

$$G(\lambda, T) = \{u \in V : (T - \lambda I)^n u = 0\}.$$

Example: If $T : \mathbf{C}^3 \to \mathbf{C}^3$ is defined by $T(x, y, z) = (y, 0, z)$, then

$$G(0, T) = \{(x, y, 0) : x, y \in \mathbf{C}\}$$
$$G(1, T) = \{(0, 0, z) : z \in \mathbf{C}\}.$$

Note that

$$\mathbf{C}^3 = G(0, T) \oplus G(1, T).$$

generalized eigenspace decomposition

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T).$$

Furthermore, each $G(\lambda_k, T)$ is invariant under T and each $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent.

The generalized eigenspace

$$G(\lambda, T) = \{u \in V : (T - \lambda I)^n u = 0\}.$$

generalized eigenspace decomposition

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T).$$

Furthermore, each $G(\lambda_k, T)$ is invariant under T and each $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent.

Proof   Every list of generalized eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent. Thus $G(\lambda_1, T) + \cdots + G(\lambda_m, T)$ is a direct sum.

This direct sum includes all generalized eigenvectors of T. Because there is a basis of V consisting of generalized eigenvectors of T (as we proved two slides ago), this implies that

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T),$$

as desired. ∎

$$G(\lambda, T) = \{u \in V : (T - \lambda I)^n u = 0\}$$

definition: *multiplicity*

Suppose λ is an eigenvalue of T. The (algebraic) *multiplicity* of λ as an eigenvalue of T is defined to be

$$\dim G(\lambda, T).$$

If λ has multiplicity d as an eigenvalue of T, then

$$(T - \lambda I)^d|_{G(\lambda,T)} = 0.$$

Because $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$, the sum of the multiplicities of all the eigenvalues of T equals n.

definition: *characteristic polynomial*

Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. The polynomial

$$q(z) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

is called the *characteristic polynomial* of T.

$$q(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}$$

Cayley–Hamilton theorem

Let q denote the characteristic polynomial of T. Then $q(T) = 0$.

block diagonal matrix with upper-triangular blocks

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where each A_k is a d_k-by-d_k upper-triangular matrix of the form

$$A_k = \begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}.$$

Reference

- *Linear Algebra Done Right*, fourth edition, Springer, 2024.

  This is an Open Access book, meaning that the electronic version is legally free to the world.

  Available at https://linear.axler.net and https://link.springer.com.

  It has been downloaded over 450,000 times.

Unlocking Hidden Dimensions:
The Power of Generalized Eigenvectors

This brilliant title was suggested by ChatGPT when I gave it the prompt "catchy title for a talk on generalized eigenvectors".