Computing with Harmonic Functions

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AMS Western Sectional Meeting

9 November 2019

references

This talk contains results from the following publications:

- Sheldon Axler and Peter J. Shin. The Neumann problem on ellipsoids, *Journal of Applied Mathematics and Computing* 57 (2018), 261–278.
- Sheldon Axler, Pamela Gorkin, and Karl Voss. The Dirichlet problem on quadratic surfaces, *Mathematics of Computation* 73 (2004), 637–651.
- Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic Function Theory*, second edition, Graduate Texts in Mathematics, Springer, 2001.
- Sheldon Axler and Wade Ramey. Harmonic polynomials and Dirichlet-type problems, *Proceedings of the American Mathematical Society* 123 (1995), 3765–3773.

Fix a positive integer n.

Suppose Ω is an open subset of $\mathbf{R}^n$.

Definition: *Laplacian*

For $u\colon \Omega \to \mathbf{R}$ in C^2, the *Laplacian* of u is denoted Δu and is the function $\Delta u\colon \Omega \to \mathbf{R}$ defined by

$$\Delta u = {D_1}^2 u + \cdots + {D_n}^2 u.$$

Definition: *harmonic*

u is called *harmonic* on Ω if

$$(\Delta u)(x) = 0$$

for all $x \in \Omega$.

examples of harmonic functions

Example: The function $u \colon \mathbf{R}^3 \to \mathbf{R}$ defined by

$$u(x_1, x_2, x_3) = {x_1}^2 + {x_2}^2 - 2{x_3}^2$$

is harmonic on $\mathbf{R}^3$.

Example: If Ω is an open subset of $\mathbf{R}^2$ and h is analytic on Ω, then $\mathrm{Re} f$ and $\mathrm{Im} f$ are harmonic on Ω.

For $x = (x_1, \dots, x_n)$, let

$$\|x\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}.$$

Example: The function

$$x \mapsto \|x\|^{2-n}$$

is harmonic on $\mathbf{R}^n \setminus \{0\}$.

Newton knew that

$$x \mapsto \frac{1}{\|x\|}$$

is harmonic on $\mathbf{R}^3 \setminus \{0\}$.

Example: If $\zeta \in \mathbf{R}^n$ and $\|\zeta\| = 1$, then

$$x \mapsto \frac{1 - \|x\|^2}{\|x - \zeta\|^n}$$

is harmonic on $\mathbf{R}^n \setminus \{\zeta\}$.

Dirichlet problem

Dirichlet problem: Suppose Ω is an open subset of $\mathbf{R}^n$. Given $f \in C(\partial\Omega)$, find $u \in C(\overline{\Omega})$ such that u is harmonic on Ω and $u|_{\partial\Omega} = f$.

Johann Dirichlet (*1805–1859*)

Let B denote the open unit ball in $\mathbf{R}^n$:

$$B = \{x \in \mathbf{R}^n : \|x\| < 1\}.$$

Thus ∂B is the unit sphere in $\mathbf{R}^n$:

$$\partial B = \{x \in \mathbf{R}^n : \|x\| = 1\}.$$

Let σ denote surface area measure on ∂B, normalized so that $\sigma(\partial B) = 1$.

solution to Dirichlet problem on B

Suppose $f \in C(\partial B)$. Define $u \colon \overline{B} \to \mathbf{R}$ by

$$u(x) = \begin{cases} \displaystyle\int_{\partial B} \frac{1 - \|x\|^2}{\|x - \zeta\|^n} f(\zeta)\, d\sigma(\zeta) & \text{if } x \in B, \\ f(x) & \text{if } x \in \partial B. \end{cases}$$

Then $u \in C(\overline{B})$ and u is harmonic on B.

The function u above is called the Poisson integral of f; it is the solution to the Dirichlet problem for f.

Poisson integral of a polynomial is a polynomial

solution to Dirichlet problem on B

Suppose $f \in C(\partial B)$. Define $u\colon \overline{B} \to \mathbf{R}$ by

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Then $u \in C(\overline{B})$ and u is harmonic on B.

Definition: $\mathcal{P}_m$

For m a positive integer, let

$\mathcal{P}_m = \{$polynomials on $\mathbf{R}^n$ with degree $\leq m\}$.

Poisson integral of a polynomial

Suppose m is a positive integer, $f \in \mathcal{P}_m$, and u is the Poisson integral of f. Then $u \in \mathcal{P}_m$ and

$$\deg u \leq \deg f.$$

Example: Suppose $n = 3$ and

$$f(x_1, x_2, x_3) = {x_1}^2 x_2 x_3.$$

Then

$$u(x) = \tfrac{1}{7}\big(x_2 x_3 - \|x\|^2 x_2 x_3 + 7{x_1}^2 x_2 x_3\big).$$

Poisson integral of a polynomial is a polynomial

Poisson integral of a polynomial

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$$\deg u \leq \deg f.$$

Proof We guess that

$$u = (1 - \|x\|^2)g + f$$

for some $g \in \mathcal{P}_{m-2}$.

We need to show that $\exists g \in \mathcal{P}_{m-2}$ with

$$\Delta\big((1 - \|x\|^2)g\big) = -\Delta f.$$

Define a linear map $L\colon \mathcal{P}_{m-2} \to \mathcal{P}_{m-2}$ by

$$Lg = \Delta\big((1 - \|x\|^2)g\big).$$

Suppose $g \in \mathcal{P}_{m-2}$ and $Lg = 0$. Thus

$$(1 - \|x\|^2)g$$

is harmonic and vanishes on ∂B. Hence

$$(1 - \|x\|^2)g = 0.$$

Thus $g = 0$. Hence L is injective.
Thus L is surjective.
Hence $\exists g \in \mathcal{P}_{m-2}$ with

$$\Delta\big((1 - \|x\|^2)g\big) = -\Delta f. \qquad \blacksquare$$

Dirichlet problem for ellipsoids

Dirichlet problem for ellipsoids

Suppose Ω is an ellipsoid in $\mathbf{R}^n$ and $f \in \mathcal{P}_m$. Then there exists a harmonic polynomial $u \in \mathcal{P}_m$ such that $u|_{\partial\Omega} = f$.

Given an ellipsoid Ω and a polynomial f, how do we find u?

None of the methods that work for the ball worked for ellipsoids.

Again, differentiation leads to a good algorithm. The results are considerably more complicated than on the ball.

Example: The harmonic polynomial that equals ${x_1}^{10}$ on

$$\{x \in \mathbf{R}^3 : 2{x_1}^2 + 3{x_2}^2 + 4{x_3}^2 = 1\}$$

has value

$$\frac{500945213823452554440546462385400584789}{3972633695067359598012898420409222215251461}$$

at the origin.

Neumann problem on ellipsoids

Suppose $\beta_1, \dots, \beta_n$ are positive numbers. Let

$$q(x) = q(x_1, \dots, x_n) = \beta_1 {x_1}^2 + \cdots + \beta_n {x_n}^2,$$

$$E = \{x \in \mathbf{R}^n : q(x) < 1\}.$$

Let $\mathbf{n}(x)$ be the outward-pointing unit normal on ∂E at $x \in \partial E$. Thus

$$\mathbf{n}(x) = \frac{\nabla q(x)}{\|\nabla q(x)\|},$$

where

$$(\nabla q)(x) = 2(\beta_1 x_1, \dots, \beta_n x_n).$$

Outward pointing normal derivative of u is

$$D_{\mathbf{n}} u = \nabla u \cdot \mathbf{n} = \nabla u \cdot \frac{\nabla q}{\|\nabla q\|}.$$

Neumann problem

Suppose $f \in \mathcal{P}_m$. Then the following are equivalent (here A = surface area):

(a) $\displaystyle\int_{\partial E} \frac{f}{\|\nabla q\|}\, dA = 0.$

(b) There exists a harmonic polynomial $u \in \mathcal{P}_m$ such that

$$\nabla u \cdot \nabla q = f \text{ on } \partial E.$$

(c) There exists a harmonic function u on $\overline{E}$ such that

$$D_{\mathbf{n}} u = \frac{f}{\|\nabla q\|} \text{ on } \partial E.$$

proof that (c) implies (a)

Neumann problem

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(c) There exists a harmonic function u on $\overline{E}$ such that

$$D_{\mathbf{n}} u = \frac{f}{\|\nabla q\|} \text{ on } \partial E.$$

A = surface area measure on ∂E.
V = volume measure on $\mathbf{R}^n$.

Green's Second Identity

If u and w are smooth functions on $\overline{E}$, then

$$\int_E (w\Delta u - u\Delta w)\, dV = \int_{\partial E} (w\, D_{\mathbf{n}} u - u\, D_{\mathbf{n}} w)\, dA.$$

Suppose u is harmonic on $\overline{E}$. Take $w \equiv 1$. Then

$$0 = \int_{\partial E} D_{\mathbf{n}} u\, dA$$

Thus (c) implies (a).

Recall that A is surface area measure on ∂E.

formula for surface area integral on ellipsoid

Suppose $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative even integers. Then

$$\int_{\partial E} \frac{x^\alpha}{\|\nabla q(x)\|}\, dA(x) = \frac{n\,\text{vol}(B)}{2\sqrt{\prod_{j=1}^n \beta_j^{\alpha_j+1}}} \cdot \frac{(\alpha_1 - 1)!! \cdots (\alpha_n - 1)!!}{n(n+2)\cdots(n+|\alpha|-2)}.$$

Kelvin transform

Definition: *inversion in the sphere*

For $x \in \mathbf{R}^n \cup \{\infty\}$, define

$$x^* = \frac{x}{\|x\|^2}.$$

Note that $(x^*)^* = x$ for all $x \in \mathbf{R}^n \cup \{\infty\}$.

If $n = 2$, then

$$z^* = \frac{z}{|z|^2} = \frac{z}{z\bar{z}} = \frac{1}{\bar{z}}.$$

Thus if u is harmonic on an open subset Ω of $\mathbf{R}^2$, then the map

$$z \mapsto u(z^*)$$

is harmonic on Ω^*.

If $n > 2$ and u is harmonic on an open subset of $\mathbf{R}^n$, then

$$x \mapsto u(x^*)$$

is almost certainly not harmonic on Ω^*.

Definition: *Kelvin transform*

Suppose u is a function defined on $\Omega \subset \mathbf{R}^n$. The *Kelvin transform* $K[u]$ is the function on Ω^* defined by

$$(K[u])(x) = \|x\|^{2-n} u(x^*).$$

$$K\big[K[u]\big] = u$$

$$K[1] = \|x\|^{2-n} \quad \text{and} \quad K\big[\|x\|^{2-n}\big] = 1$$

Kelvin transform and harmonic functions

$$(K[u])(x) = \|x\|^{2-n} u\left(\frac{x}{\|x\|^2}\right)$$

Kelvin transform preserves harmonicity

- A function is harmonic if and only if its Kelvin transform is harmonic.
- More precisely, suppose $u \colon \Omega \to \mathbf{R}$ is defined on an open set $\Omega \subset \mathbf{R}^n$. Then u is harmonic on Ω if and only if $K[u]$ is harmonic on Ω^*.

Laplacian of Kelvin transform

If u is a C^2 function on an open subset of $\mathbf{R}^n \setminus \{0\}$, then

$$\Delta\big(K[u]\big) = K\big[\|x\|^4 \Delta u\big].$$

Lord Kelvin (*1824–1907*)

using Kelvin transform to compute Poisson integral

$$D = (D_1, \ldots, D_n)$$

Example: If $f(x_1, x_2, x_3) = 5{x_1}^2{x_2}^3{x_3}^8$, then

$$f(D)u = 5{D_1}^2{D_2}^3{D_3}^8u.$$

mapping from polynomials to harmonic polynomials

If $f \in \mathcal{P}_m$, then $K[f(D)\|x\|^{2-n}]$ is a harmonic polynomial in $\mathcal{P}_m$.

Let

$$c_m = \prod_{k=0}^{m-1}(2 - n - 2k).$$

Poisson integral formula

If f is a homogeneous polynomial in $\mathcal{P}_m$, then the Poisson integral of f equals

$$\frac{1}{c_m}K[f(D)\|x\|^{2-n}] + g$$

for some polynomial $g \in \mathcal{P}_{m-2}$.

The result above leads to a very fast algorithm for computing the Poisson integral of a polynomial.