# Compact Operators via the Berezin Transform

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### Abstract

In this paper we prove that if $S$ equals a finite sum of finite products of Toeplitz operators on the Bergman space of the unit disk, then $S$ is compact if and only if the Berezin transform of $S$ equals 0 on $\partial D$. This result is new even when $S$ equals a single Toeplitz operator. Our main result can be used to prove, via a unified approach, several previously known results about compact Toeplitz operators, compact Hankel operators, and appropriate products of these operators.

## 1　Introduction

Let $dA$ denote Lebesgue area measure on the unit disk $D$, normalized so that the measure of $D$ equals 1. The *Bergman space* $L_a^2$ is the Hilbert space consisting of the analytic functions on $D$ that are also in $L^2(D, dA)$. For $z \in D$, the *Bergman reproducing kernel* is the function $K_z \in L_a^2$ such that

$$f(z) = \langle f, K_z \rangle$$

for every $f \in L_a^2$. The *normalized Bergman reproducing kernel* $k_z$ is the function $K_z / \|K_z\|_2$. Here, as elsewhere in this paper, the norm $\| \ \|_2$ and the inner product $\langle \ , \ \rangle$ are taken in the space $L^2(D, dA)$.

For $S$ a bounded operator on $L_a^2$, the *Berezin transform* of $S$ is the function $\tilde{S}$ on $D$ defined by

$$\tilde{S}(z) = \langle S k_z, k_z \rangle.$$

For $u \in L^\infty(D, dA)$, the *Toeplitz operator* $T_u$ with symbol $u$ is the operator on $L_a^2$ defined by $T_u f = P(uf)$, where $P$ is the orthogonal projection from $L^2(D, dA)$ onto $L_a^2$.

In this paper we prove that if $S$ equals a finite sum of finite products of Toeplitz operators, then $S$ is compact if and only if $\tilde{S}(z) \to 0$ as $z \to \partial D$. This result is new even when $S$ equals a single Toeplitz operator $T_u$. Our main result can be used to prove, via a unified approach, several previously known results about compact Toeplitz operators, compact Hankel operators, and appropriate products of these operators.

A common intuition is that for operators on the Bergman space "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on $\partial D$. Our main result shows that this intuition is correct if "closely associated with function theory" is interpreted to mean that the operator is a finite sum of finite products of Toeplitz operators.

Section 2 of this paper contains a precise statement of our theorem along with a discussion of some consequences and examples. Section 3 contains three lemmas that will be used in the proof of the theorem. Section 4 contains the proof of the theorem.

## 2   Discussion of the Theorem

A nice survey of previously known results connecting the Berezin transform with Toeplitz operators (on both the Hardy space and the Bergman space) can be found in [8].

Before stating our theorem, we need to introduce some notation. For $z \in D$, let $\varphi_z$ be the analytic map of $D$ onto $D$ defined by

$$(2.1) \qquad \varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

A simple computation shows that $\varphi_z \circ \varphi_z$ is the identity function on $D$.

For $z \in D$, let $U_z \colon L_a^2 \to L_a^2$ be the unitary operator defined by

$$U_z f = (f \circ \varphi_z)\varphi_z{}'.$$

Notice that $U_z{}^* = U_z{}^{-1} = U_z$, so $U_z$ is actually a self-adjoint unitary operator.

For $S$ a bounded operator on $L_a^2$, define $S_z$ to be the bounded operator on $L_a^2$ given by conjugation with $U_z$:

$$S_z = U_z S U_z.$$

Although we are concerned only with operators on the Bergman space $L_a^2$, we will need to make use of other norms. For a measurable function $u$ on $D$ and $1 \leq p \leq \infty$, let $\|u\|_p$ denote the usual norm of $u$ in $L^p(D, dA)$.

Now we are ready to state our main result. Although we will discuss some consequences of this theorem in this section, we will not complete the proof of this theorem until Section 4.

**Theorem 2.2** *Suppose $S$ is a finite sum of operators of the form $T_{u_1} \ldots T_{u_n}$, where each $u_j \in L^\infty(D, dA)$. Then the following are equivalent:*

(a)  *$S$ is compact;*

(b)  *$\|Sk_z\|_2 \to 0$ as $z \to \partial D$;*

(c)  *$\tilde{S}(z) \to 0$ as $z \to \partial D$;*

(d)  *$S_z 1 \to 0$ weakly in $L_a^2$ as $z \to \partial D$;*

(e)  *$\|S_z 1\|_2 \to 0$ as $z \to \partial D$;*

(f)  *$\|S_z 1\|_p \to 0$ as $z \to \partial D$ for every $p \in (1, \infty)$.*

Our main concern is the equivalence of conditions (a) and (c) above, but the other conditions are also of interest. Furthermore, the other conditions are needed as intermediate steps in the proof that (c) implies (a). One curious feature of the proof is that even though we are only interested in the Bergman space $L_a^2$, we will need to use condition (f) with $p = 6$ as part of the chain of implications from (c) to (a), as the reader will note when we present the proof in Section 4.

The Berezin transform $\tilde{u}$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator $T_u$. In other words, $\tilde{u} = \widetilde{T_u}$. Note that $\tilde{u}(z) = \widetilde{T_u}(z) = \langle T_u k_z, k_z \rangle = \langle P(uk_z), k_z \rangle = \langle uk_z, k_z \rangle$ for each $z \in D$. Expressing this as an integral, we obtain the formula

$$(2.3) \qquad \tilde{u}(z) = \int_D u(w)|k_z(w)|^2 \, dA(w)$$

for each $z \in D$. Because $\|k_z\|_2 = 1$, this formula shows that $\tilde{u}(z)$ is a weighted average of $u$.

As is well known, the explicit formulas for the reproducing kernel and the normalized reproducing kernel are given by

$$(2.4) \qquad K_z(w) = \frac{1}{(1 - \bar{z}w)^2}, \quad k_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2}$$

for $z, w \in D$. Along with the formula (2.1) for $\varphi_z$, this shows that $|\varphi_z{}'(w)| = |k_z(w)|$ for all $z, w \in D$. Thus making the change of variables $\lambda = \varphi_z(w)$

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in (2.3), we have $dA(\lambda) = |k_z(w)|^2\, dA(w)$ and $w = \varphi_z(\lambda)$ (because $\varphi_z$ is its own inverse under composition), transforming (2.3) into the formula

$$\tilde{u}(z) = \int_D (u \circ \varphi_z)(\lambda)\, dA(\lambda)$$

for each $z \in D$.

The corollary below is an immediate consequence of the equivalence of conditions (a) and (c) in Theorem 2.2. Previously, the best results known along these lines were special cases that required additional hypotheses on $u$. Specifically, Zhu ([13], Theorem B) proved the result below under the additional hypothesis that $u$ is a nonnegative function. More recently, Korenblum and Zhu [6] proved the result below under the additional hypothesis that $u$ is a radial function (meaning $u(z) = u(|z|)$ for all $z \in D$) and Stroethoff [9] proved it under the additional hypothesis that $u$ is uniformly continuous with respect to the hyperbolic metric.

**Corollary 2.5** *If $u \in L^\infty(D, dA)$, then $T_u$ is compact if and only if $\tilde{u}(z) \to 0$ as $z \to \partial D$.*

Now we turn to a discussion of how Theorem 2.2 can be used prove several known results about Toeplitz and Hankel operators. The advantage of using Theorem 2.2 is that it allows a single technique to be used in several different contexts, replacing more specialized techniques and estimates that might work only in one context.

We begin with a characterization of the compact Toeplitz operators. Actually Corollary 2.5 above provides the most useful characterization of the compact Toeplitz operators, in the sense that the condition $\tilde{u}(z) \to 0$ as $z \to \partial D$ is easier to verify in practice than the conditions we will discuss below. However, the other conditions below are also useful, and we want to demonstrate how Theorem 2.2 gives proofs of these results.

Suppose $u \in L^\infty(D, dA)$. Zheng proved ([12], Theorem 4) that the following are equivalent:

(i)  $T_u$ is compact;

(ii)  $\|T_u k_z\|_2 \to 0$ as $z \to \partial D$;

(iii)  $\|P(u \circ \varphi_z)\|_2 \to 0$ as $z \to \partial D$;

(iv)  $\|P(u \circ \varphi_z)\|_p \to 0$ as $z \to \partial D$ for every $p \in (1, \infty)$.

To prove this using Theorem 2.2, let $S = T_u$. The equivalence of (i) and (ii) above is a special case of the equivalence of conditions (a) and (b) in Theorem 2.2. An easy calculation (see Lemma 8 of [3]) shows that $S_z = T_{u \circ \varphi_z}$, so $S_z 1 = P(u \circ \varphi_z)$. Thus the equivalence of (i), (iii), and (iv) above follows from the equivalence of conditions (a), (e), (f) in Theorem 2.2, completing our proof of Zheng's result.

For the next application of Theorem 2.2, we need a formula for $S_z$ when $S$ is a finite product of Toeplitz operators. If $u_1, \ldots, u_n \in L^\infty(D, dA)$, then

$$(2.6) \qquad U_z T_{u_1} \ldots T_{u_n} U_z = T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z}$$

because we can write the operator on the left side as

$$(U_z T_{u_1} U_z)(U_z T_{u_2} U_z) \ldots (U_z T_{u_n} U_z)$$

and then use the formula $U_z T_u U_z = T_{u \circ \varphi_z}$ (see Lemma 8 of [3]).

Now we show how the compact Hankel operators can be characterized by using Theorem 2.2. Again suppose that $u \in L^\infty(D, dA)$. The *Hankel operator* with symbol $u$ is the operator $H_u$ from $L_a^2$ to $L^2(D, dA) \ominus L_a^2$ defined by $H_u f = (1 - P)(uf)$.

Stroethoff and Zheng proved ([8], Theorem 6, and [12], Theorem 3) that the following are equivalent:

(i)    $H_u$ is compact;

(ii)    $\|H_u k_z\|_2 \to 0$ as $z \to \partial D$;

(iii)    $\|u \circ \varphi_z - P(u \circ \varphi_z)\|_2 \to 0$ as $z \to \partial D$.

To prove this using Theorem 2.2, let $S = H_u{}^* H_u$. We need to work with $H_u{}^* H_u$ instead of $H_u$ because $H_u$ is not a finite sum of finite products of Toeplitz operators. However, the definitions of Hankel and Toeplitz operators easily lead to the identity

$$(2.7) \qquad H_u{}^* H_u = T_{|u|^2} - T_{\bar{u}} T_u,$$

so Theorem 2.2 can be applied to $S$. The equivalence of conditions (a) and (c) in Theorem 2.2 shows that $H_u{}^* H_u$ is compact (which is equivalent to $H_u$ being compact) if and only if $(H_u{}^* H_u)\tilde{}(z) \to 0$ as $z \to \partial D$. However

$$(H_u{}^* H_u)\tilde{}(z) = \langle H_u{}^* H_u k_z, k_z \rangle = \|H_u k_z\|_2{}^2,$$

so we conclude that $H_u$ is compact if and only if $\|H_u k_z\|_2 \to 0$ as $z \to \partial D$. In other words, conditions (i) and (ii) above are equivalent. To get to condition (iii), note that from (2.6) and (2.7), we have $S_z = H_{u \circ \varphi_z}{}^* H_{u \circ \varphi_z}$. The

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equivalence of conditions (a) and (b) in Theorem 2.2 thus shows that $H_u$ is compact if and only if $\|H_{u\circ\varphi_z}{}^* H_{u\circ\varphi_z} 1\|_2 \to 0$ as $z \to \partial D$. Because

$$\|H_{u\circ\varphi_z} 1\|_2{}^2 \leq \|H_{u\circ\varphi_z}{}^* H_{u\circ\varphi_z} 1\|_2 \leq \|H_{u\circ\varphi_z}{}^*\|\|H_{u\circ\varphi_z} 1\|_2,$$

we see that $H_u$ is compact if and only if $\|H_{u\circ\varphi_z} 1\|_2 \to 0$ as $z \to \partial D$. Finally, note that $H_{u\circ\varphi_z} 1 = u \circ \varphi_z - P(u \circ \varphi_z)$. Thus conditions (i) and (iii) above are equivalent, completing our proof of Stroethoff's and Zheng's result.

As a special case of the last result, suppose $u = \bar{f}$, where $f$ is a bounded analytic function on $D$. Then $P(\bar{f} \circ \varphi_z)$ equals the constant function $\bar{f}(z)$. Thus from the equivalence of conditions (i) and (iii) above we conclude that $H_{\bar{f}}$ is compact if and only if $\|f \circ \varphi_z - f(z)\|_2 \to 0$ as $z \to \partial D$. This last condition holds if and only if $f$ is in the little Bloch space (see Theorem 2 of [2]). Thus we have recovered Axler's result ([2], Theorem 7) that $H_{\bar{f}}$ is compact if and only if $f$ is in the little Bloch space.

We emphasize that the proofs just given of the results due to Zheng, Stroethoff and Zheng, and Axler should not be regarded as entirely new proofs of these results. Some techniques from the original proofs (along with some techniques used by Stroethoff and Zheng in [11]) have been incorporated into part of our proof of Theorem 2.2 (specifically, some of these techniques are used in our proof that condition (f) of Theorem 2.2 implies condition (a) of Theorem 2.2). The techniques just mentioned could be used to prove that conditions (a) and (b) in Theorem 2.2 are equivalent, but they do not seem powerful enough to prove that conditions (a) and (c) are equivalent (this is our main goal). Thus the proof of Theorem 2.2 requires some additional techniques, which we introduce in Section 4.

Our paper [4] gives two additional applications of Theorem 2.2, including a characterization of the bounded analytic functions $f, g$ such that $T_f T_{\bar{g}} - T_{\bar{g}} T_f$ is compact.

Now we turn to a discussion of the role of the hypothesis of Theorem 2.2, which requires that $S$ be a finite sum of finite products of Toeplitz operators. Two of the trivial implications in Theorem 2.2, namely (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c), obviously hold for arbitrary bounded operators $S$ on $L_a^2$. An examination of the proof (see Section 4) shows that one of the difficult implications, namely (c) $\Rightarrow$ (d), also holds for arbitrary bounded operators $S$ on $L_a^2$. The proofs of the other implications all use the hypothesis that $S$ is a finite sum of finite products of Toeplitz operators.

The paragraph above suggests that we should consider whether Theorem 2.2 holds for arbitrary bounded operators $S$ on $L_a^2$. Are the key conditions (a), (b), and (c) of Theorem 2.2 equivalent for arbitrary bounded

operators $S$ on $L_a^2$? Unfortunately this question has a negative answer. In fact, we will now give some examples to show that no two of the conditions (a), (b), and (c) of Theorem 2.2 are equivalent for arbitrary bounded operators $S$ on $L_a^2$. Thus the hypothesis that $S$ is a finite sum of finite products of Toeplitz operators (or some appropriate substitute) is needed in Theorem 2.2.

For our examples we will need a power series formula for the Berezin transform of a bounded operator $S$ on $L_a^2$. From (2.4) we get

$$k_z(w) = (1 - |z|^2) \sum_{m=0}^{\infty} (m+1) \bar{z}^m w^m$$

for $z, w \in D$. To compute $\tilde{S}(z)$, which equals $\langle Sk_z, k_z \rangle$, first compute $Sk_z$ by applying $S$ to both sides of the equation above, and then take the inner product with $k_z$, again using the equation above, to obtain

$$(2.8) \qquad \tilde{S}(z) = (1 - |z|^2)^2 \sum_{m,n=0}^{\infty} (m+1)(n+1) \langle Sw^m, w^n \rangle \bar{z}^m z^n.$$

Consider the operator $S$ on $L_a^2$ defined by

$$S\left( \sum_{n=0}^{\infty} a_n w^n \right) = \sum_{n=0}^{\infty} a_{2^n} w^{2^n}.$$

Clearly $S$ is a self-adjoint projection with infinite-dimensional range. Thus $S$ is not compact. Furthermore,

$$\begin{aligned} \|Sk_z\|_2{}^2 &= \langle Sk_z, k_z \rangle \\ &= \tilde{S}(z) \\ &= (1 - |z|^2)^2 \sum_{n=0}^{\infty} (2^n + 1)(|z|^2)^{2^n}, \end{aligned}$$

where the first equality holds because $S$ is a self-adjoint projection and the last equality follows from (2.8). Because $(1-t)^2 \sum_{n=0}^{\infty} (2^n + 1) t^{2^n} \to 0$ as $t \to 1^-$ (we leave the verification of this limit as an exercise), the equation above shows that $\|Sk_z\|_2{}^2 = \tilde{S}(z) \to 0$ as $z \to \partial D$. This shows that neither condition (b) nor condition (c) of Theorem 2.2 implies condition (a) for arbitrary operators on $L_a^2$.

Now consider the operator $S$ on $L_a^2$ defined by

$$S\left( \sum_{n=0}^{\infty} a_n w^n \right) = \sum_{n=0}^{\infty} (-1)^n a_n w^n.$$

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Clearly $S$ is a unitary operator, so $\|Sk_z\|_2 = 1$ for all $z \in D$. Thus $\|Sk_z\|_2 \not\to 0$ as $z \to \partial D$. However, a calculation (use (2.8)) shows that

$$\tilde{S}(z) = \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2}$$

for $z \in D$. Thus $\tilde{S}(z) \to 0$ as $z \to \partial D$. Hence $S$ satisfies condition (c) of Theorem 2.2 but not condition (b), showing that condition (c) does not imply condition (b) for arbitrary operators on $L_a^2$. This example is based on a similar example provided by Nordgren and Rosenthal [7] in the context of Hardy spaces.

After this paper appeared in preprint form, Miroslav Engliš (private communication) showed that a conjectured quantitative improvement of Theorem 2.2 is false. Specifically, he shows that the essential norm of $T_u$ and $\limsup_{z \to \partial D} |\tilde{u}(z)|$ are not equivalent for $u \in L^\infty(D, dA)$.

## 3   Some Lemmas

This section contains three lemmas that will be used in the proof of Theorem 2.2. We begin by stating a simple lemma describing how the Berezin transform acts under composition with the maps $\varphi_z$. Recall that $K_z$ denotes the reproducing kernel on $L_a^2$ and that $k_z$ denotes the normalized reproducing kernel $K_z / \|K_z\|_2$.

**Lemma 3.1** *If $S$ is a bounded operator on $L_a^2$ and $z \in D$, then $\tilde{S} \circ \varphi_z = \widetilde{S_z}$.*

The proof of the lemma above is a calculation that we leave to the reader as an exercise.

Note that $(S_w)^* = (S^*)_w$ for every bounded operator $S$ on $L_a^2$ and every $w \in D$. Thus the expression $S_w^*$, which appears in (3.4) below, can be interpreted either way.

For $S$ a bounded operator on $L_a^2$ and $z, w \in D$, the function $S_z 1$ or the function $S_w^* 1$ may not be in $L^6(D, dA)$. In that case, the right side of (3.3) or (3.4) below will be infinity, making the corresponding inequality trivially true.

The appearance of the $L^6$ norm in the next lemma may seem strange. However, the lemma becomes false if the 6 is replaced by 2, because otherwise we would be able to prove that conditions (a) and (b) in Theorem 2.2 are equivalent for arbitrary bounded operators $S$ on $L_a^2$ (which we have already shown is false).

**Lemma 3.2** *There is a constant $c < \infty$ such that if $S$ is a bounded operator on $L_a^2$, then*

$$(3.3) \qquad \int_D \frac{|(SK_z)(w)|}{\sqrt{1-|w|^2}}\, dA(w) \le \frac{c\|S_z 1\|_6}{\sqrt{1-|z|^2}}$$

*for all $z \in D$ and*

$$(3.4) \qquad \int_D \frac{|(SK_z)(w)|}{\sqrt{1-|z|^2}}\, dA(z) \le \frac{c\|S_w^* 1\|_6}{\sqrt{1-|w|^2}}$$

*for all $w \in D$.*

PROOF: An easy calculation shows that

$$(3.5) \qquad U_z 1 = (|z|^2 - 1)K_z.$$

To prove (3.3), fix a bounded operator $S$ on $L_a^2$ and fix $z \in D$. We have

$$SK_z = \frac{SU_z 1}{|z|^2 - 1} = \frac{U_z S_z 1}{|z|^2 - 1} = \frac{((S_z 1) \circ \varphi_z)\varphi_z{}'}{|z|^2 - 1},$$

where the first equality comes from (3.5), the second equality comes from the definition of $S_z$, and the third equality comes from the definition of $U_z$. Thus

$$\int_D \frac{|(SK_z)(w)|}{\sqrt{1-|w|^2}}\, dA(w) = \frac{1}{1-|z|^2} \int_D \frac{|(S_z 1)(\varphi_z(w))|\, |\varphi_z{}'(w)|}{\sqrt{1-|w|^2}}\, dA(w).$$

In the last integral, make the substitution $w = \varphi_z(\lambda)$ and use the identities

$$\varphi_z(\varphi_z(\lambda)) = \lambda$$

$$\varphi_z{}'(\varphi_z(\lambda)) = \frac{1}{\varphi_z{}'(\lambda)} = \frac{(1-\bar{z}\lambda)^2}{|z|^2 - 1}$$

$$(3.6) \qquad 1 - |\varphi_z(\lambda)|^2 = \frac{(1-|z|^2)(1-|\lambda|^2)}{|1-\bar{z}\lambda|^2}$$

$$dA(w) = |\varphi_z{}'(\lambda)|^2\, dA(\lambda) = \frac{(1-|z|^2)^2}{|1-\bar{z}\lambda|^4}\, dA(\lambda)$$

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to obtain

$$\int_D \frac{|(SK_z)(w)|}{\sqrt{1-|w|^2}}\, dA(w) = \frac{1}{\sqrt{1-|z|^2}} \int_D \frac{|(S_z 1)(\lambda)|}{|1-\bar{z}\lambda|\sqrt{1-|\lambda|^2}}\, dA(\lambda).$$

Hölder's inequality now gives

$$\int_D \frac{|(SK_z)(w)|}{\sqrt{1-|w|^2}}\, dA(w) \leq \frac{\|S_z 1\|_6}{\sqrt{1-|z|^2}} \left( \int_D \frac{dA(\lambda)}{|1-\bar{z}\lambda|^{6/5}(1-|\lambda|^2)^{3/5}} \right)^{5/6}.$$

Lemma 4 of Axler's paper [2] asserts that there exists a constant $c < \infty$ independent of $z$ (and of course also independent of $S$) such that the last integral on the right is bounded by $c$. This completes the proof of (3.3).

To prove (3.4), replace $S$ with $S^*$ in (3.3), interchange the roles of $w$ and $z$ in (3.3) and then use the equation

$$(3.7) \qquad (S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, SK_z \rangle = \overline{(SK_z)(w)}$$

to obtain the desired result. ∎

The next lemma is the final tool needed for the proof of Theorem 2.2.

**Lemma 3.8** *If $S$ is a finite sum of operators of the form $T_{u_1} \ldots T_{u_n}$, where each $u_j \in L^\infty(D, dA)$, then*

$$\sup_{z \in D} \|S_z 1\|_p < \infty$$

*for every $p \in (1, \infty)$.*

PROOF: By the triangle inequality, we can assume without loss of generality that $S = T_{u_1} \ldots T_{u_n}$, where each $u_j \in L^\infty(D, dA)$. Using (2.6) and the definition of $S_z$, we have

$$(3.9) \qquad S_z = T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z}.$$

Fix $p \in (1, \infty)$. Then there is a constant $c < \infty$ such that $\|Pv\|_p \leq c\|v\|_p$ for every $v \in L^p(D, dA)$ (see [1], Theorem 1.10). Thus $\|T_u f\|_p \leq c\|u\|_\infty \|f\|_p$ for all $u \in L^\infty(D, dA), f \in L_a^2$. Because $\|u \circ \varphi_z\|_\infty$ is independent of $z$, this implies that $\|T_{u \circ \varphi_z} f\|_p \leq c\|u\|_\infty \|f\|_p$ for all $u \in L^\infty(D, dA)$, all $f \in L_a^2$, and all $z \in D$. This, along with (3.9), shows that $\|S_z 1\|_p$ is bounded independent of $z$, as desired. ∎

## 4   The Proof

Now we are ready to prove Theorem 2.2. We will show that

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a).$$

Several of these implications are easy or trivial. The difficult parts of the proof are the implications (c) $\Rightarrow$ (d) and (f) $\Rightarrow$ (a).

PROOF OF THEOREM 2.2:   First suppose that (a) holds, so $S$ is compact. Because $k_z \to 0$ weakly in $L_a^2$ as $z \to \partial D$, this implies that $\|Sk_z\|_2 \to 0$ as $z \to \partial D$, completing the proof that (a) implies (b).

Now suppose that (b) holds, so $\|Sk_z\|_2 \to 0$ as $z \to \partial D$. Then

$$|\tilde{S}(z)| = |\langle Sk_z, k_z \rangle| \leq \|Sk_z\|_2.$$

Hence $\tilde{S}(z) \to 0$ as $z \to \partial D$, completing the proof that (b) implies (c).

Now suppose that (c) holds, so $\tilde{S}(z) \to 0$ as $z \to \partial D$. To prove (d), it suffices to show that $\langle S_z 1, w^n \rangle \to 0$ as $z \to \partial D$ for every nonnegative integer $n$. So fix a nonnegative integer $n$.

For $z, \lambda \in D$, we have

$$\tilde{S}(\varphi_z(\lambda)) = \widetilde{S_z}(\lambda)$$
$$= (1 - |\lambda|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1)\langle S_z w^j, w^m \rangle \bar{\lambda}^j \lambda^m,$$

where the first equality comes from Lemma 3.1 and the second equality comes from (2.8). Now fix $r \in (0,1)$, multiply both sides of the last equation by $\bar{\lambda}^n/(1 - |\lambda|^2)^2$, and then integrate over $rD$ to obtain

$$\int_{rD} \frac{\tilde{S}(\varphi_z(\lambda))\bar{\lambda}^n}{(1-|\lambda|^2)^2} \, dA(\lambda) = \sum_{j,m=0}^{\infty} (j+1)(m+1)\langle S_z w^j, w^m \rangle \int_{rD} \bar{\lambda}^{j+n} \lambda^m \, dA(\lambda)$$
$$= \sum_{j=0}^{\infty} (j+1)\langle S_z w^j, w^{j+n} \rangle r^{2j+2n+2}$$
$$= r^{2n+2} \left( \langle S_z 1, w^n \rangle + \sum_{j=1}^{\infty} (j+1)\langle S_z w^j, w^{j+n} \rangle r^{2j} \right).$$

For each $\lambda \in D$, we know that $\varphi_z(\lambda) \to \partial D$ as $z \to \partial D$ (this follows from (3.6)). Thus by our hypothesis (c), $\tilde{S}(\varphi_z(\lambda)) \to 0$ as $z \to \partial D$ for each

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$\lambda \in D$. Hence for each fixed $r \in (0, 1)$, the left side of the equality above has limit 0 as $z \to \partial D$ (note that the integrand is bounded by $\|S\|/(1 - r^2)^2$, independently of $\lambda$ and $z$, justifying our passage to of the limit). Dividing the right side of the last equality above by $r^{2n+2}$ (we are thinking of $r$ as fixed), we conclude that

$$(4.1) \qquad \langle S_z 1, w^n \rangle + \sum_{j=1}^{\infty} (j+1) \langle S_z w^j, w^{j+n} \rangle r^{2j} \to 0 \text{ as } z \to \partial D$$

for each $r \in (0, 1)$.

Let's examine the infinite sum above. Note that

$$(4.2) \qquad \left| \sum_{j=1}^{\infty} (j+1) \langle S_z w^j, w^{j+n} \rangle r^{2j} \right| \le \|S\| \sum_{j=1}^{\infty} r^{2j}$$
$$= \|S\| \frac{r^2}{1 - r^2}.$$

Thus given $\epsilon > 0$, we can choose $r \in (0, 1)$ such that the left side of (4.2) is less than $\epsilon$ for all $z \in D$. Hence (4.1) implies that

$$\limsup_{z \to \partial D} |\langle S_z 1, w^n \rangle| \le \epsilon.$$

Because $\epsilon$ is an arbitrary positive number, the inequality above implies that $\langle S_z 1, w^n \rangle \to 0$ as $z \to \partial D$, as desired, completing the proof that (c) implies (d).

Now suppose that (d) holds, so $S_z 1 \to 0$ weakly in $L_a^2$ as $z \to \partial D$. For every $z \in D$ and every $r \in (0, 1)$ we have

$$\|S_z 1\|_2{}^2 = \int_{D \setminus r\bar{D}} |(S_z 1)(w)|^2 \, dA(w) + \int_{r\bar{D}} |(S_z 1)(w)|^2 \, dA(w)$$

$$(4.3) \qquad \le (1 - r^2)^{1/2} \|S_z 1\|_4{}^2 + \int_{r\bar{D}} |(S_z 1)(w)|^2 \, dA(w),$$

where the inequality comes from writing $|(S_z 1)(w)|^2 = 1|(S_z 1)(w)|^2$ and then using the Cauchy-Schwarz inequality in the first integral above. By Lemma 3.8, $\|S_z 1\|_4$ is bounded independent of $z$. Thus given $\epsilon > 0$, we can choose $r \in (0, 1)$ such that the first term in (4.3) is less than $\epsilon/2$ for all $z \in D$. Having chosen $r \in (0, 1)$, the second term in (4.3) will also be less than $\epsilon/2$ for all $z$ sufficiently close to $\partial D$ (because a sequence converging weakly to 0 in $L_a^2$ converges uniformly to 0 on each compact subset of $D$).

Thus for $z$ sufficiently close to $\partial D$ we have $\|S_z 1\|_2^2 < \epsilon$, completing the proof that (d) implies (e).

Now suppose that (e) holds, so

$$(4.4) \qquad \|S_z 1\|_2 \to 0 \text{ as } z \to \partial D.$$

To prove (f), fix $p \in (1, \infty)$. If $1 < p \le 2$, then clearly (4.4) implies that $\|S_z 1\|_p \to 0$ as $z \to \partial D$. To consider the remaining case, suppose now that $2 < p < \infty$. Then

$$(4.5) \qquad \|S_z 1\|_p \le \|S_z 1\|_2^{1/p} \|S_z 1\|_{2p-2}^{(p-1)/p},$$

as can be seen by writing $|S_z 1|^p = |S_z 1||S_z 1|^{p-1}$ and then using the Cauchy-Schwarz inequality in the integral defining $\|S_z 1\|_p$. By our hypothesis (e), the first term on the right side of (4.5) has limit $0$ as $z \to \partial D$. The second term on the right side of (4.5) is bounded independent of $z$, by Lemma 3.8. Thus the left side of (4.5) has limit $0$ as $z \to \partial D$, completing the proof that (e) implies (f).

Now suppose that (f) holds, so $\|S_z 1\|_6 \to 0$ as $z \to \partial D$. For $f \in L_a^2$ and $w \in D$, we have

$$
\begin{aligned}
(Sf)(w) &= \langle Sf, K_w \rangle \\
&= \langle f, S^* K_w \rangle \\
&= \int_D f(z) \overline{(S^* K_w)(z)} \, dA(z) \\
(4.6) \qquad &= \int_D f(z)(SK_z)(w) \, dA(z),
\end{aligned}
$$

where the last equation follows from (3.7).

For $0 < r < 1$, define an operator $S_{[r]}$ on $L_a^2$ by

$$(4.7) \qquad (S_{[r]} f)(w) = \int_{rD} f(z)(SK_z)(w) \, dA(z).$$

In other words, $S_{[r]}$ is the integral operator with kernel $(SK_z)(w)\chi_{rD}(z)$. For each $r \in (0, 1)$ we have

$$
\begin{aligned}
\int_D \int_D |(SK_z)(w)\chi_{rD}(z)|^2 \, dA(w) \, dA(z) &= \int_{rD} \int_D |SK_z(w)|^2 \, dA(w) \, dA(z) \\
&= \int_{rD} \|SK_z\|_2^2 \, dA(z) \\
&\le \|S\|^2 \int_{rD} \|K_z\|_2^2 \, dA(z) \\
&< \infty.
\end{aligned}
$$

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Thus $S_{[r]}$ is a Hilbert-Schmidt operator and in particular is compact. Hence to prove that (f) implies (a), we only need show that $\|S - S_{[r]}\| \to 0$ as $r \to 1^-$.

If $r \in (0,1)$, then $S - S_{[r]}$ is the integral operator with kernel

$$(SK_z)(w)\chi_{D \setminus rD}(z),$$

as can be seen from (4.6) and (4.7). The Schur test (see page 126 of [5]) implies that if $u$ is a positive measurable function on $D$ and $c_1, c_2$ are constants such that

$$(4.8) \qquad \int_D |(SK_z)(w)\chi_{D \setminus rD}(z)|u(w)\,dA(w) \le c_1 u(z)$$

for all $z \in D$ and

$$(4.9) \qquad \int_D |(SK_z)(w)\chi_{D \setminus rD}(z)|u(z)\,dA(z) \le c_2 u(w)$$

for all $w \in D$, then

$$(4.10) \qquad \|S - S_{[r]}\| \le \sqrt{c_1 c_2}.$$

Note that the left side of (4.8) equals 0 for $0 < |z| < r$. Taking $u(\lambda) = 1/\sqrt{1 - |\lambda|^2}$, we see from Lemma 3.2 that (4.8) and (4.9) are satisfied with $c_1 = c \sup\{\|S_z 1\|_6 : r \le |z| < 1\}$ and $c_2 = c \sup\{\|S_w^* 1\|_6 : w \in D\}$, where $c$ is the constant from Lemma 3.2. Our hypothesis (f) implies that $c_1 \to 0$ as $r \to 1^-$, and Lemma 3.8 (with $S^*$ replacing $S$) shows that $c_2 < \infty$. Thus from (4.10) we conclude that $\|S - S_{[r]}\| \to 0$ as $r \to 1^-$, completing the proof that (f) implies (a). ∎

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