Playing Nice Together: Commuting Operators

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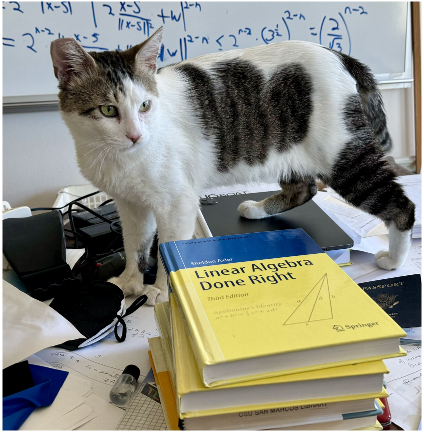
6 January 2026



Standing assumptions for this talk:

- V is a finite-dimensional nonzero complex vector space; $n = \dim V$.
- $\mathcal{L}(V)$ denotes the set of linear maps from V to V (operators).

definition: *commuting operators*

- Two operators $S, T \in \mathcal{L}(V)$ are said to *commute* if
$$ST = TS.$$
- Two square matrices A and B of the same size are said to *commute* if
$$AB = BA.$$

- There are 674,609 ordered pairs of 2-by-2 commuting matrices all of whose entries are integers in $[-5, 5]$.
- Approximately one out of every 318 ordered pairs of 2-by-2 matrices all of whose entries are integers in $[-5, 5]$ commute.

Haiku on commuting operators:

Operators
pass in either order—
the same result waits.

–ChatGPT

Example: Suppose m is a positive integer and V is the complex vector space of polynomials (with coefficients in $\mathbf{C}$) in two real variables and of degree at most m.

Thus the elements of V are functions p from $\mathbf{R}^2$ to $\mathbf{C}$ of the form

$$p(x, y) = \sum_{j+k \leq m} a_{j,k}\, x^j y^k,$$

where the indices j and k take on all nonnegative integer values such that $j + k \leq m$ and each $a_{j,k}$ is in $\mathbf{C}$.

Define operators $D_x, D_y \in \mathcal{L}(V)$ by

$$D_x p = \frac{\partial p}{\partial x} = \sum_{j+k \leq m} j a_{j,k}\, x^{j-1} y^k$$

and

$$D_y p = \frac{\partial p}{\partial y} = \sum_{j+k \leq m} k a_{j,k}\, x^j y^{k-1}.$$

Then D_x and D_y commute because

$$(D_x D_y)p = \sum_{j+k \leq m} jk a_{j,k}\, x^{j-1} y^{k-1}$$

$$= (D_y D_x)p.$$

definition: ***eigenvalue***

Suppose $S \in \mathcal{L}(V)$ and $\lambda \in \mathbf{C}$.

- λ is called an *eigenvalue* of S if $S - \lambda I$ is not injective, which happens if and only if $\operatorname{null}(S - \lambda I) \neq \{0\}$.
- A nonzero vector $u \in V$ is called an *eigenvector* of S corresponding to λ if $Su = \lambda u$.

existence of eigenvalues

Every operator on V has an eigenvalue.

Example: Suppose m is a positive integer and

$$V = \Big\{ \sum_{j+k \leq m} a_{j,k}\, x^j y^k \Big\}.$$

0 is the only eigenvalue of D_x; the set of eigenvectors of D_x is

$$\Big\{ \sum_{k=0}^{m} b_k\, y^k \Big\} \setminus \{0\}.$$

0 is also the only eigenvalue of D_y; the set of eigenvectors of D_y is

$$\Big\{ \sum_{j=0}^{m} c_j\, x^j \Big\} \setminus \{0\}.$$

null S is invariant under commuting T

Suppose $S, T \in \mathcal{L}(V)$ commute.
Then $\operatorname{null} S$ is invariant under T.

Proof Suppose $u \in \operatorname{null} S$. Then

$$S(Tu) = T(Su) = T(0) = 0.$$

Thus $Tu \in \operatorname{null} S$.
Hence $\operatorname{null} S$ is invariant under T. ■

Example: The nonzero constant functions are common eigenvectors of D_x and D_y from the previous example.

commuting implies a common eigenvector

Every pair of commuting operators on V has a common eigenvector.

Proof Suppose $S, T \in \mathcal{L}(V)$ commute. Let λ be an eigenvalue of S. Because $S - \lambda I$ and T commute, $T|_{\operatorname{null}(S-\lambda I)}$ maps $\operatorname{null}(S - \lambda I)$ into itself (by first result on this slide).

The operator $T|_{\operatorname{null}(S-\lambda I)}$ has an eigenvector, which is a common eigenvector for S and T. ■

An upper-triangular matrix:

$$\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

If $S \in \mathcal{L}(V)$, there is an orthogonal basis of V with respect to which S has an upper-triangular matrix.

simultaneous upper-triangularization

If $S, T \in \mathcal{L}(V)$ commute, then there is a basis of V with respect to which both S and T have upper-triangular matrices.

Proof is by induction on $n = \dim V$.

When necessary, add the assumption that V is an inner product space.

commuting version of Schur's theorem

If S and T are commuting operators on V, then there is an orthonormal basis of V with respect to which both S and T have upper-triangular matrices.

Proof Suppose $u_1, \ldots, u_n$ is a basis of V with respect to which S and T have upper-triangular matrices. Use G.S. to get an orthonormal basis $e_1, \ldots, e_n$ of V with

$$\operatorname{span}(e_1, \ldots, e_k) = \operatorname{span}(u_1, \ldots, u_k)$$

for $k = 1, \ldots, n$. ■

definition: *normal operator*

An operator is called *normal* if it commutes with its adjoint.

spectral theorem

Suppose $S \in \mathcal{L}(V)$ is normal. Then there exists an orthonormal basis of V consisting of eigenvectors of S.

Proof By the commuting version of Schur's theorem, there is an orthonormal basis of V with respect to which both S and S^* have upper-triangular matrices.

$$\mathcal{M}(S) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The matrix of S^* with respect to this orthonormal basis is the conjugate transpose of the matrix of S.

Thus

$$\mathcal{M}(S) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Hence we have an orthonormal basis consisting of eigenvectors of S. ∎

eigenvalues of sum and product

Suppose $S, T \in \mathcal{L}(V)$ commute. Then

- every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T;
- every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.

Let $\sigma(S) = \{\text{eigenvalues of } S\}$. If $S, T \in \mathcal{L}(V)$ commute, then

$$\sigma(S+T) \subseteq \sigma(S) + \sigma(T),$$

$$\sigma(ST) \subseteq \sigma(S) \cdot \sigma(T).$$

Proof There exists a basis of V such that

$$\mathcal{M}(S) = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \mathcal{M}(T) = \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ 0 & & \beta_n \end{pmatrix},$$

where $\sigma(S) = \{\lambda_1, \dots, \lambda_n\}$ and $\sigma(T) = \{\beta_1, \dots, \beta_n\}$.

Thus

$$\mathcal{M}(S+T) = \begin{pmatrix} \lambda_1 + \beta_1 & & * \\ & \ddots & \\ 0 & & \lambda_n + \beta_n \end{pmatrix}.$$

Hence

$$\sigma(S+T) = \{\lambda_1 + \beta_1, \dots, \lambda_n + \beta_n\}.$$

Similarly,

$$\sigma(ST) = \{\lambda_1\beta_1, \dots, \lambda_n\beta_n\}. \quad \blacksquare$$

self-adjoint products

The product of two self-adjoint operators on V is self-adjoint if and only if the two operators commute.

Proof Suppose $S, T \in \mathcal{L}(V)$ are self-adjoint operators. Then

$$(ST)^* = T^* S^* = TS.$$

Thus $(ST)^* = ST \iff ST = TS$. ∎

definition: *positive operator*

An operator is called *positive* if it is self-adjoint and all its eigenvalues are nonnegative numbers.

positive products

The product of two positive operators on V is positive if and only if the two operators commute.

Proof Suppose $S, T \in \mathcal{L}(V)$ are positive and commute. Recall that

$$\sigma(ST) \subseteq \sigma(S) \cdot \sigma(T).$$

Thus all the eigenvalues of ST are nonnegative numbers. ∎

definition: *diagonalizable*

- A *diagonal matrix* is a square matrix that is 0 everywhere except possibly on the diagonal.
- $S \in \mathcal{L}(V)$ is called *diagonalizable* if S has a diagonal matrix with respect to some basis of V.

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

$S \in \mathcal{L}(V)$ is diagonalizable if and only if there exists a basis of V consisting of eigenvectors of S.

simultaneous diagonalizability

Suppose S and T are diagonalizable. Then the following are equivalent:

- S and T have diagonal matrices with respect to the same basis of V.
- S and T commute.

simultaneous diagonal & upper-triangular

Suppose $S, T \in \mathcal{L}(V)$ commute and S is diagonalizable. Then there is a basis of V with respect to which S has a diagonal matrix and T has an upper-triangular matrix.

eigenvalues of S

Suppose $S \in \mathcal{L}(V)$. Then

1. there exists $\lambda \in \mathbf{C}$ such that $\operatorname{null}(S - \lambda I) \neq \{0\}$;
2. there exists $\lambda \in \mathbf{C}$ such that $\operatorname{range}(S - \lambda I) \neq V$;
3. there exists $\lambda \in \mathbf{C}$ such that
$$D(S - \lambda I) \neq I$$
for all $D \in \mathcal{L}(V)$;
4. there exists $\lambda \in \mathbf{C}$ such that
$$(S - \lambda I)D \neq I$$
for all $D \in \mathcal{L}(V)$.

common eigenvectors of commuting S, T

Suppose $S, T \in \mathcal{L}(V)$ commute. Then

1. there exists $\lambda, \beta \in \mathbf{C}$ such that $\operatorname{null}(S - \lambda I) \cap \operatorname{null}(T - \beta I) \neq \{0\}$;
2. there exist $\lambda, \beta \in \mathbf{C}$ such that $\operatorname{range}(S - \lambda I) + \operatorname{range}(T - \beta I) \neq V$;
3. there exist $\lambda, \beta \in \mathbf{C}$ such that
$$D(S - \lambda I) + E(T - \beta I) \neq I$$
for all $D, E \in \mathcal{L}(V)$;
4. there exist $\lambda, \beta \in \mathbf{C}$ such that
$$(S - \lambda I)D + (T - \beta I)E \neq I$$
for all $D, E \in \mathcal{L}(V)$.

definition: *eigenvalue pairs*

An ordered pair $(\lambda, \beta) \in \mathbf{C}^2$ is called an *eigenvalue pair* of (S, T) if

$$\operatorname{null}(S - \lambda I) \cap \operatorname{null}(T - \beta I) \neq \{0\}.$$

A nonzero vector $u \in V$ is called an *eigenvector* of (S, T) corresponding to (λ, β) if

$$Su = \lambda u \quad \text{and} \quad Tu = \beta u.$$

existence of eigenvalue pairs

Suppose $S, T \in \mathcal{L}(V)$ commute. If λ is an eigenvalue of S, then $\exists\, \beta \in \mathbf{C}$ such that (λ, β) is an eigenvalue pair of (S, T).

linearly independent eigenvectors

Suppose $S, T \in \mathcal{L}(V)$. Then every list of eigenvectors of (S, T) corresponding to distinct eigenvalue pairs of (S, T) is linearly independent.

at most n eigenvalue pairs of (S, T)

If $S, T \in \mathcal{L}(V)$, then (S, T) has at most n distinct eigenvalue pairs.

common eigenvector for families of commuting operators

Suppose $\mathcal{F} \subseteq \mathcal{L}(V)$ is such that S and T commute for all $S, T \in \mathcal{F}$. Then:

1. there is a vector in V that is a common eigenvector for all operators in $\mathcal{F}$;
2. there is a basis of V with respect to which every operator in $\mathcal{F}$ has an upper-triangular matrix.

If V is an inner product space, then the basis in part 2 above can be taken to be an orthonormal basis.

commuting spectral theorem for families of normal operators

Suppose $\mathcal{F} \subseteq \mathcal{L}(V)$ is a family of commuting normal operators on V. Then there is an orthonormal basis of V with respect to which every operator in $\mathcal{F}$ has a diagonal matrix.

commuting spectral theorem for families of normal operators

Suppose $\mathcal{F} \subseteq \mathcal{L}(V)$ is a family of commuting normal operators on V. Then there is an orthonormal basis of V with respect to which every operator in $\mathcal{F}$ has a diagonal matrix.

Suppose W is a finite-dimensional real inner product space.

commuting spectral theorem for families of self-adjoint operators

Suppose $\mathcal{F} \subseteq \mathcal{L}(W)$ is a family of commuting self-adjoint operators on W. Then there is an orthonormal basis of W with respect to which every operator in $\mathcal{F}$ has a diagonal matrix.

definition: *generalized eigenvector*

Suppose $S \in \mathcal{L}(V)$. A nonzero vector $u \in V$ is called a *generalized eigenvector* of S corresponding to an eigenvalue λ of S if

$$(S - \lambda I)^n u = 0.$$

Note: $T^k u = 0$ for some $k \iff T^n u = 0$.

Example: Suppose $S \in \mathcal{L}(\mathbf{C}^3)$ is defined by

$$S(z_1, z_2, z_3) = (z_2, 0, z_3).$$

The eigenvectors corresponding to the eigenvalue 0 are nonzero vectors of the form $(z_1, 0, 0)$.

The generalized eigenvectors of S corresponding to the eigenvalue 0 are nonzero vectors of the form $(z_1, z_2, 0)$.

The eigenvectors (and the generalized eigenvectors) corresponding to the eigenvalue 1 are nonzero vectors of the form $(0, 0, z_3)$.

basis of generalized eigenvectors

Suppose $S \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of S.

basis of generalized eigenvectors

Suppose $S \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of S.

commuting version

Suppose $S, T \in \mathcal{L}(V)$ commute. Then there is a basis of V consisting of vectors that are generalized eigenvectors of both S and T.

Proof If λ is an eigenvalue of S, then $\operatorname{null}(S - \lambda I)^n$ is invariant under T.

Thus $\exists$ a basis of $\operatorname{null}(S - \lambda I)^n$ consisting of the generalized eigenvectors of $T|_{\operatorname{null}(S-\lambda I)^n}$.

Put these together to get a basis of V consisting of generalized eigenvectors of both S and T. ∎

definition: ***nilpotent operator***

An operator $S \in \mathcal{L}(V)$ is called *nilpotent* if

$$S^k = 0$$

for some positive integer k.

nilpotent depends only on n^{th} power

Suppose $S \in \mathcal{L}(V)$. Then

$$S \text{ is nilpotent} \iff S^n = 0.$$

sum & product of commuting nilpotents

Suppose $S, T \in \mathcal{L}(V)$ are commuting nilpotent operators. Then ST and $S + T$ are nilpotent.

Proof To show that ST is nilpotent, note that

$$(ST)^n = S^n T^n = 0.$$

To show $S + T$ is nilpotent, note that

$$(S+T)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} S^{2n-k} T^k$$
$$= 0. \quad \blacksquare$$

For $S \in \mathcal{L}(V)$ and $\lambda \in \mathbf{C}$, let

$$G(\lambda, S) = \{u \in V : (S - \lambda I)^n u = 0\}.$$

generalized eigenvector decomposition

Suppose $S \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of S. Then

- $V = G(\lambda_1, S) \oplus \cdots \oplus G(\lambda_m, S)$;
- $\dim V = \sum_{j=1}^{m} \dim G(\lambda_j, S)$;
- S maps each $G(\lambda_j, S)$ into $G(\lambda_j, S)$;
- each $(S - \lambda_j I)|_{G(\lambda_j, S)}$ is nilpotent.

commuting version of gen. e. decomposition

Suppose $S, T \in \mathcal{L}(V)$ commute. Let $(\lambda_1, \beta_1), \ldots, (\lambda_m, \beta_m)$ be the distinct eigenvalue pairs of (S, T). Then

- $V = G(\lambda_1, \beta_1, S, T) \oplus \cdots \oplus G(\lambda_m, \beta_m, S, T)$;
- $\dim V = \sum_{j=1}^{m} \dim G(\lambda_j, \beta_j, S, T)$;
- both S and T map each $G(\lambda_j, \beta_j, S, T)$ into $G(\lambda_j, \beta_j, S, T)$;
- each $(S - \lambda_j I)|_{G(\lambda_j, \beta_j, S, T)}$ is nilpotent;
- each $(T - \beta_j I)|_{G(\lambda_j, \beta_j, S, T)}$ is nilpotent.

For $S, T \in \mathcal{L}(V)$ and $\lambda, \beta \in \mathbf{C}$, let

$$G(\lambda, \beta, S, T) = \{u \in V : (S - \lambda I)^n u = (T - \beta I)^n u = 0\} = G(\lambda, S) \cap G(\beta, T).$$

nilpotent operator is not surjective

Suppose $S \in \mathcal{L}(V)$ is nilpotent. Then S is not surjective; in other words

$$\operatorname{range} S \neq V.$$

commuting nilpotent operators

Suppose $S, T \in \mathcal{L}(V)$ are commuting nilpotent operators. Then

$$\operatorname{range} S + \operatorname{range} T \neq V.$$

Proof Simultaneous upper-triangularization for commuting operators implies that there exists a basis $e_1, \ldots, e_n$ of V such that $\mathcal{M}(S)$ and $\mathcal{M}(T)$ each have the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

Thus

$$Su + Tw \neq e_n$$

for all $u, w \in V$. Hence

$$\operatorname{range} S + \operatorname{range} T \neq V. \quad \blacksquare$$

injectivity $\iff$ *surjectivity*

Suppose $S \in \mathcal{L}(V)$. Then

$$\operatorname{null} S = \{0\} \iff \operatorname{range} S = V.$$

injectivity $\iff$ *surjectivity, commuting version*

Suppose $S, T \in \mathcal{L}(V)$ commute. Then

$$\operatorname{null} S \cap \operatorname{null} T = \{0\} \iff \operatorname{range} S + \operatorname{range} T = V.$$

Proof: Let $U = G(0, 0, S, T) = G(0, S) \cap G(0, T)$.

$\Leftarrow$: Suppose $\operatorname{null} S \cap \operatorname{null} T \neq \{0\}$. Thus $U \neq 0$.

A previous result showed that

$$\operatorname{range} S|_U + \operatorname{range} T|_U \neq U.$$

Because each $G(\lambda, \beta, S, T)$ is invariant under both S and T, this implies that

$$\operatorname{range} S + \operatorname{range} T \neq V.$$

$\Rightarrow$: Now suppose $U = \{0\}$. Thus for each $G(\lambda, \beta, S, T) \neq \{0\}$, either $\lambda \neq 0$ or $\beta \neq 0$. Thus either $S|_{G(\lambda,\beta,S,T)}$ or $T|_{G(\lambda,\beta,S,T)}$ is invertible for every nonzero $G(\lambda, \beta, S, T)$. Hence

$$\operatorname{range} S + \operatorname{range} T = V. \qquad \blacksquare$$

Jordan form

Suppose $S \in \mathcal{L}(V)$. Then there is a basis of V with respect to which the matrix of S is a Jordan matrix.

Is there a commuting version of the Jordan form theorem:

Suppose $S, T \in \mathcal{L}(V)$ commute. Is there a basis of V with respect to which both S and T have Jordan matrices?

Answer: Not necessarily.

Fuglede's theorem (1950, PNAS)

Suppose $S, T \in \mathcal{L}(V)$ commute and S is normal. Then S commutes with T^*.

Proof There is an orthonormal basis of V such that the matrix of S with respect to this basis is a diagonal matrix.

Let $\lambda_1, \ldots, \lambda_n$ be the diagonal entries of this matrix (these are the eigenvalues of S).

Let p be a polynomial such that

$$p(\lambda_k) = \overline{\lambda_k}$$

for $k = 1, \ldots, \lambda_n$.

Then $S^* = p(S)$ because the matrix of S^* and the matrix of $p(S)$ both equal the diagonal matrix with $\overline{\lambda_1}, \ldots, \overline{\lambda_n}$ on the diagonal.

Because T commutes with $p(S)$, we conclude that T commutes with S^*.

Take adjoints of both sides of the equation $TS^* = S^*T$ to conclude that $ST^* = T^*S$. ■

Reference

- *Linear Algebra Done Right*, fourth edition, Springer, 2024.

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