A Stroll through Bergman Space

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D = open unit disk

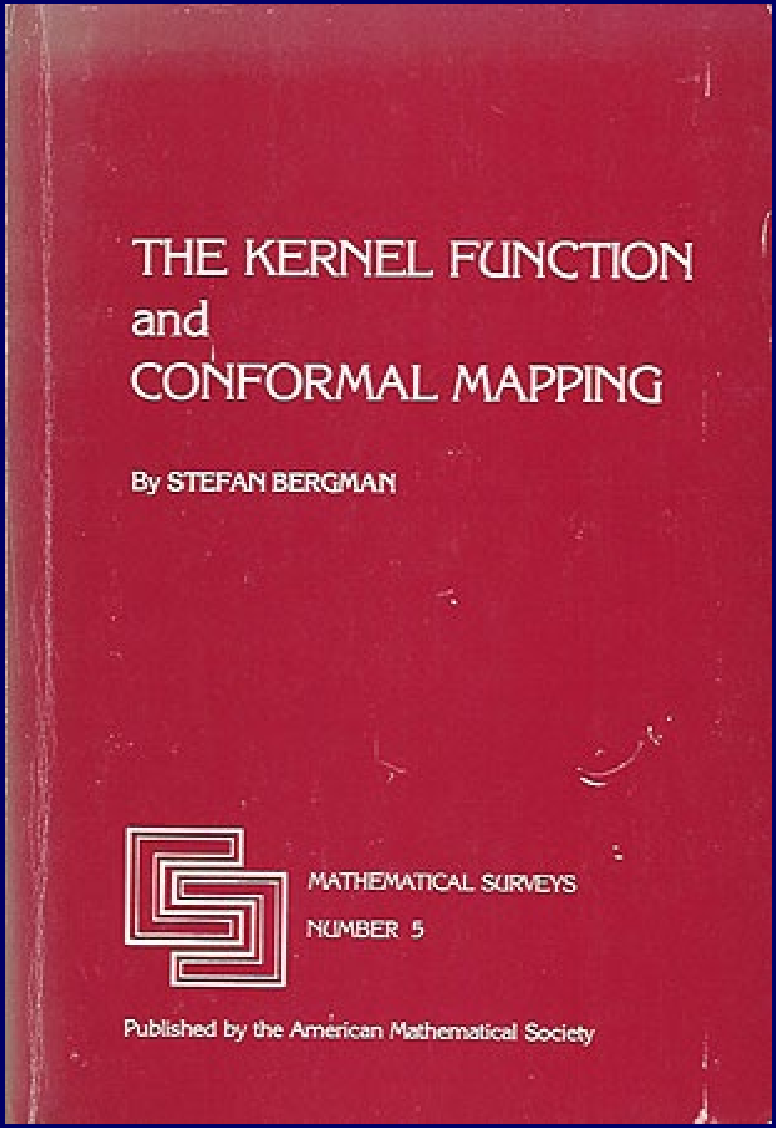
dA = normalized area

$1 \leq p < \infty$

Bergman space $B^p =$

$\{h$ analytic on D :

$$||h||_p = \left(\int_D |h|^p \, dA \right)^{1/p} < \infty \}$$



Bergman space $B^p =$

$$\{h \text{ analytic on } D : \|h\|_p = \Big(\int_D |h|^p \, dA\Big)^{1/p} < \infty\}$$

Hardy space $H^p =$

$\{h$ analytic on D :

$$\|h\|_p = \sup_{0<r<1} \Big(\int_0^{2\pi} |h(re^{i\theta})|^p \frac{d\theta}{2\pi}\Big)^{1/p} < \infty\}$$

$H^p \subset B^p$, clearly.

$H^p \subset B^{2p}$, which is considerably deeper.

$h(z) = \sum_{n=0}^{\infty} a_n z^n \Rightarrow$

$$\text{the } B^2 \text{ norm of } h = \Big(\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} \Big)^{1/2}$$

and

$$\text{the } H^2 \text{ norm of } h = \Big(\sum_{n=0}^{\infty} |a_n|^2 \Big)^{1/2}.$$

$\{a_n\} \subset D$ is a B^p *zero set* if $\exists h \in B^p$ such that

$$\{a_n\} = h^{-1}(0).$$

$\{a_n\}$ is an H^p zero set $\Longleftrightarrow$

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$$

Horowitz (1974): $\{a_n\}$ is a B^p zero set $\Rightarrow$

$$\sum_{n=1}^{\infty} (1 - |a_n|)^{1+\epsilon} < \infty$$

for all $\epsilon > 0$.

Horowitz (1974): If $1 \leq p < q < \infty$, then there exists a B^p zero set that is not a B^q zero set.

Horowitz (1974): The union of two B^p zero sets is not necessarily a B^p zero set.

Horowitz (1974): Every subset of a B^p zero set is a B^p zero set.

Proof: Let $h \in B^p$ and $\{a_n\} \subset h^{-1}(0)$. Then

$$\frac{h(z)}{\prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z} \left(2 - \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}\right)} \in B^p.$$

A closed subspace $E \subset B^2$ is called *invariant* if

$$zE \subset E.$$

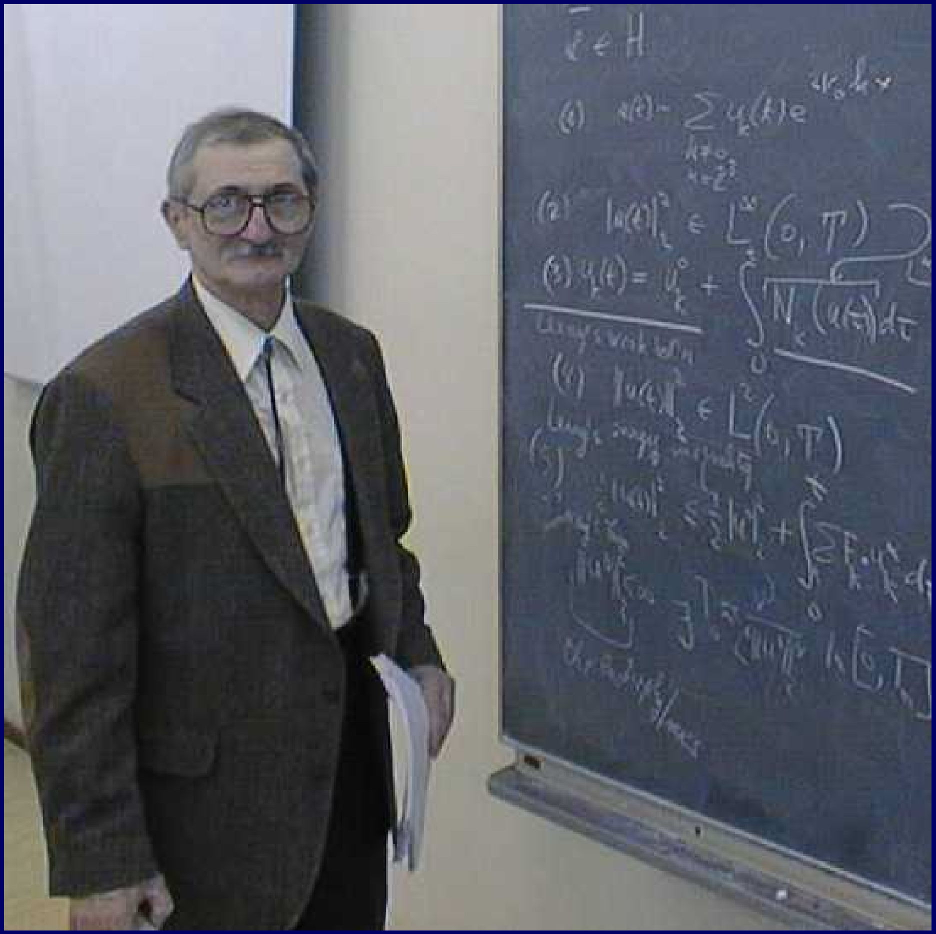
Example: Fix $\{a_n\} \subset D$. Let

$$E = \{h \in B^2 : h(a_n) = 0 \quad \forall n\}.$$

Corollary: There exist two nonzero invariant subspaces of B^2 whose intersection is $\{0\}$.

If $E \subset H^2$ is invariant, then $\dim E/zE = 1$.

Apostol, Bercovici, Foias, Pearcy (1985): Let $n \in \{1, 2, \ldots\} \cup \{\infty\}$. Then there exists an invariant subspace $E \subset B^2$ such that $\dim E/zE = n$.



Ciprian Foias

Let g be a bounded analytic function on D. Define $T_g h = gh$.

Nehari (1957): ${T_g}^* T_g - T_g {T_g}^*$ is compact on H^2

$$\iff \bar{g} \in H^\infty + C$$

$$\iff g \in \mathit{VMO}.$$

Axler (1986): ${T_g}^* T_g - T_g {T_g}^*$ is compact on B^2

$$\iff \lim_{|z| \to 1} (1 - |z|) g'(z) = 0.$$

Suppose $\{a_n\}$ is an H^2 zero sequence.

Let

$$b(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}.$$

If $h \in H^2$, $\|h\|_2 = 1$, and $h(a_n) = 0 \ \forall n$, then

$$|h(0)| = \left| \frac{h}{b}(0) \ b(0) \right| \leq b(0).$$

Thus

$$b(0) = \sup\{|h(0)| : h \in H^2, \|h\|_2 = 1, h(a_n) = 0 \ \forall n\}.$$

Suppose $\{a_n\}$ is a B^2 zero sequence.

Let b be the unique function with $b(0) > 0$ that attains the sup in

$$\sup\{|h(0)| : h \in B^2, \|h\|_2 = 1, h(a_n) = 0\ \forall n\}.$$

Example: If $\{a_n\}$ contains only one point a, then

$$b(z) = \frac{1}{|a|\sqrt{2 - |a|^2}} \left[1 - \left(\frac{1 - |a|^2}{1 - \overline{a}z} \right)^2 \right].$$

Hedenmalm (1991):

Let b be the solution to the extremal problem for functions zero on $\{a_n\}$.

Then $\{a_n\}$ is the zero set of b.

Furthermore, if $h \in B^2$ and $h(a_n) = 0 \ \forall n$, then

$$\|\frac{h}{b}\|_2 \leq \|h\|_2.$$

Haakan Hedenmalm

Suppose $E \subset B^2$ is invariant. The *extremal function* for E is the unique function b with $b(0) > 0$ that attains the sup in

$$\sup\{|h(0)| : h \in E, \|h\|_2 = 1\}.$$

$b \perp \{h \in E : h(0) = 0\}$; thus $b \perp zE$;

thus $b \perp z^n b \ \forall n > 0$; thus

$$(*) \qquad \int_D |b(z)|^2 z^n \, dA(z) = 0 \quad \forall n > 0.$$

Functions $b \in B^2$ with $\|b\|_2 = 1$ satisfying (*) are called B^2-*inner*.

For $A \subset B^2$, let $[A] =$ smallest closed invariant subspace of B^2 containing A.

Aleman, Richter, Sundberg (1996): If $E \subset B^2$ is invariant and $\dim E/zE = 1$, then

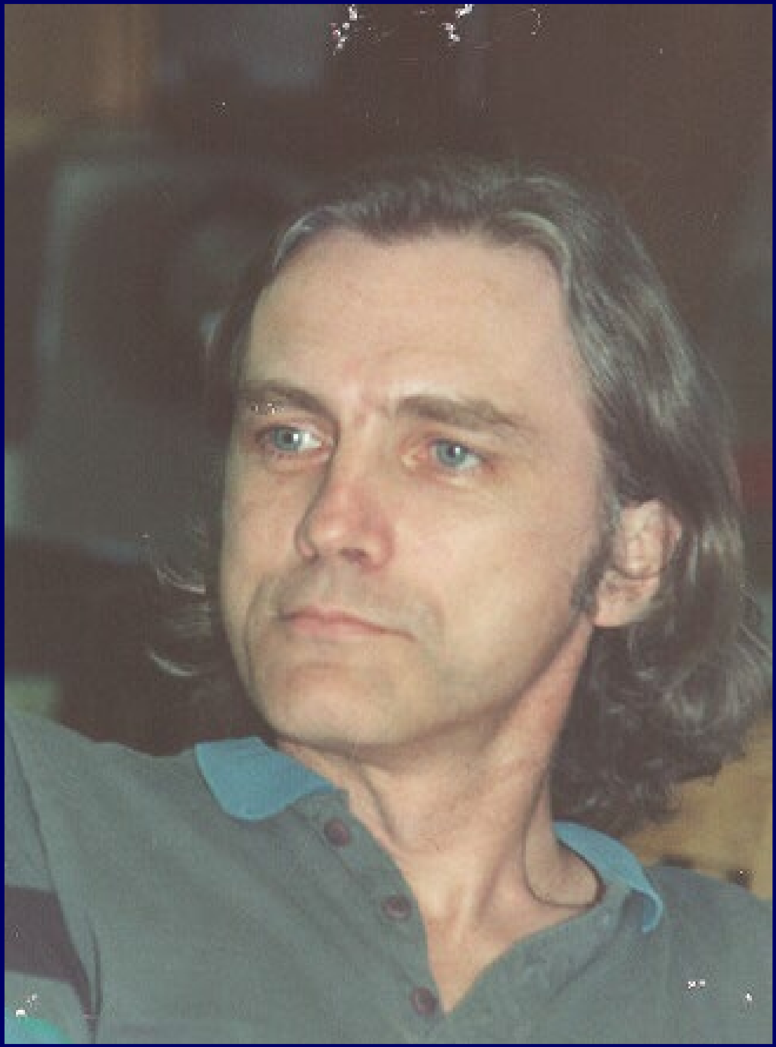
$$E = [b],$$

where b is the extremal B^2-inner function.

Stefan Richter

Aleman, Richter, Sundberg (1996): If $E \subset B^2$ is invariant, then

$$E = [E \ominus zE].$$



Carl Sundberg

$k \in B^2$ is a B^2-*outer function* if $|g(0)| \leq |k(0)|$ whenever $g \in B^2$ and $\|pg\|_2 \leq \|pk\|_2$ for all polynomials p.

Aleman, Richter, Sundberg (1996): Let $k \in B^2$. Then $[k] = B^2$ if and only if k is B^2-outer.

Aleman, Richter, Sundberg (1996): Let $h \in B^2$. Then there exist a B^2-inner function b and a B^2-outer function k such that

$$h = bk.$$

P = orthogonal projection from $L^2(D, dA)$ onto B^2.

For $g \in L^\infty(D, dA)$, define Toeplitz operator $T_g : B^2 \to B^2$ by

$$T_g h = P(gh).$$

For which $g \in L^\infty(D, dA)$ is T_g compact?

Otto Toeplitz

For $z \in D$, there exists $K_z \in B^2$ such that

$$h(z) = \langle h, K_z \rangle \quad \forall h \in B^2.$$

$$\frac{K_z(w)}{\|K_z\|_2} = \frac{1 - |z|^2}{(1 - \overline{z}w)^2}$$

For $S : B^2 \to B^2$, define the Berezin transform $\tilde{S} : D \to \mathbf{C}$ by

$$\tilde{S}(z) = \left\langle S\Big(\frac{K_z}{\|K_z\|_2}\Big), \frac{K_z}{\|K_z\|_2} \right\rangle.$$

For $g \in L^\infty(D, dA)$, let $\tilde{g} = \widetilde{T_g}$.

$$\tilde{g}(z) = \left\langle T_g\left(\frac{K_z}{\|K_z\|_2}\right), \frac{K_z}{\|K_z\|_2} \right\rangle$$

$$= \left\langle \frac{gK_z}{\|K_z\|_2}, \frac{K_z}{\|K_z\|_2} \right\rangle$$

$$= \int_D g(w) \frac{(1 - |z|^2)^2}{|1 - \overline{z}w|^4} dA(w)$$

If g is analytic, then $\tilde{g} = g$.

If g is harmonic, then $\tilde{g} = g$.

Ahern, Flores, Rudin (1993): If $g \in L^1(D, dA)$ and $\tilde{g} = g$, then g is harmonic.

$$\tilde{S}(z) = \left\langle S\left(\frac{K_z}{\|K_z\|_2}\right), \frac{K_z}{\|K_z\|_2} \right\rangle$$

If S is compact, then

$$\tilde{S}(z) \to 0 \quad \text{as } |z| \to 1,$$

but the converse is not true.

Axler, Zheng (1998): Suppose S is a finite sum of finite products of Toeplitz operators. Then S is compact if and only if

$$\tilde{S}(z) \to 0 \quad \text{as } |z| \to 1.$$

Stephan Bergman