Bôcher's Theorem

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INTRODUCTION. Bôcher's Theorem characterizes the behavior of positive harmonic functions in the neighborhood of an isolated singularity. Let n denote a positive integer greater than 1. Recall that a real-valued function u defined on an open set $\Omega \subset \mathbf{R}^n$ is said to be *harmonic* in Ω if u is twice continuously differentiable and

$$\Delta u \equiv 0$$

in Ω, where

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

Let B_n denote the open unit ball in $\mathbf{R}^n$. If $n = 2$, the function $\log(1/|x|)$ is positive and harmonic in $B_2 \setminus \{0\}$, while if $n > 2$, the function $|x|^{2-n}$ is positive and harmonic in $B_n \setminus \{0\}$. Bôcher's Theorem illustrates how important these particular functions are:

Bôcher's Theorem. *Suppose u is positive and harmonic in $B_n \setminus \{0\}$. Then there exists a function v harmonic in B_n and a constant $a \geqslant 0$ such that*

(i) $u(x) = a\log(1/|x|) + v(x)$ *for all* $x \in B_2 \setminus \{0\}$ (*if* $n = 2$);

(ii) $u(x) = a|x|^{2-n} + v(x)$ *for all* $x \in B_n \setminus \{0\}$ (*if* $n > 2$).

The usual proofs of Bôcher's Theorem rely either on the theory of superharmonic functions ([**4**], Theorem 5.4) or series expansions using spherical harmonics ([**5**], Chapter X, Theorem XII). (The referee has called our attention to the proof given by G. E. Raynor [**7**]. Raynor points out that the original proof of Maxime Bôcher [**2**] implicitly uses some non-obvious properties of the level surfaces of a harmonic function.) In this paper we offer a different and simpler approach to this theorem. The only results about harmonic functions needed are the minimum principle, Harnack's Inequality, and the solvability of the Dirichlet problem in B_n.

We will investigate a harmonic function by studying its dilates. For u a function defined on $B_n \setminus \{0\}$ and $r \in (0, 1)$, the *dilate* u_r is the function defined on $(1/r)B_n \setminus \{0\}$ by

$$u_r(x) = u(rx).$$

Note that every dilate of a harmonic function is harmonic.

For convenience, we assume in the rest of this paper that $n > 2$; all statements and proofs will easily carry over to the $n = 2$ case (with $\log(1/|x|)$ in place of $|x|^{2-n}$).

SPHERICAL AVERAGES. Let S denote the unit sphere in $\mathbf{R}^n$. Given a continuous function u defined in $B_n \setminus \{0\}$, we define $A[u](x)$, the average of u over the sphere of radius $|x|$, by

$$A[u](x) = \frac{1}{\sigma(S)} \int_S u(|x|\zeta)\, d\sigma(\zeta) \qquad (x \in B_n \setminus \{0\});$$

here σ denotes surface-area measure.

The following lemma is well known to potential theorists. The elementary proof given here was suggested by the referee.

Lemma 1. *If u is harmonic in $B_n \setminus \{0\}$, then there are constants a and b such that*

$$A[u](x) = a|x|^{2-n} + b$$

for all $x \in B_n \setminus \{0\}$. In particular, $A[u]$ is harmonic in $B_n \setminus \{0\}$.

Proof: Define f on $(0, 1)$ by

$$f(r) = \frac{1}{\sigma(S)} \int_S u(r\zeta)\, d\sigma(\zeta);$$

so $A[u](x) = f(|x|)$. Because u is continuously differentiable on $B_n \setminus \{0\}$, we can compute f' by differentiating under the integral sign, obtaining

$$f'(r) = \frac{1}{\sigma(S)} \int_S \zeta \cdot (\nabla u)(r\zeta)\, d\sigma(\zeta) = \frac{r^{-n}}{\sigma(S)} \int_{rS} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau).$$

Let $0 < r_0 < r_1 < 1$, and let $\Omega = \{x \in \mathbf{R}^n\colon r_0 < |x| < r_1\}$. The divergence theorem, applied to ∇u, shows that

$$\int_{\partial\Omega} \mathbf{n} \cdot (\nabla u)(\tau)\, d\sigma(\tau) = \int_\Omega (\Delta u)(\tau)\, dV(\tau);$$

here $\mathbf{n}$ denotes the outward unit normal on $\partial\Omega$, σ denotes surface-area measure on $\partial\Omega$, and V denotes Lebesgue volume measure on $\mathbf{R}^n$. Because u is harmonic on Ω, the right hand side of the equation above is 0. Note also that $\partial\Omega = r_0 S \cup r_1 S$ and that $\mathbf{n} = -\tau/r_0$ on $r_0 S$ and $\mathbf{n} = \tau/r_1$ on $r_1 S$. Thus the equation above implies that

$$\frac{1}{r_0} \int_{r_0 S} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau) = \frac{1}{r_1} \int_{r_1 S} \tau \cdot (\nabla u)(\tau)\, d\sigma(\tau),$$

which means $f'(r)$ is a constant multiple of r^{1-n} (for $0 < r < 1$). Hence $f(r)$ is of the form $ar^{2-n} + b$, as desired. ■

Remark. Lemma 1 shows that every radial harmonic function in $B_n \setminus \{0\}$ has the form $a|x|^{2-n} + b$ (a function is called *radial* if its value at x depends only on $|x|$).

Lemma 2. *There exists a positive constant α such that for every positive harmonic u in $B_n \setminus \{0\}$,*

$$\alpha u(y) < u(x) \text{ whenever } 0 < |x| = |y| \leqslant 1/2.$$

Proof: Harnack's Inequality (see [4], Theorem 2.16) states that if Ω is a connected open subset of $\mathbf{R}^n$ and K is a compact subset of Ω, then there is a positive constant α such that

$$\alpha u(y) < u(x)$$

for every positive harmonic function u in Ω and all $x, y \in K$. Thus there exists $\alpha > 0$ such that for all positive harmonic u in $B_n \setminus \{0\}$, $\alpha u(y) < u(x)$ whenever $|x| = |y| = 1/2$. Applying this result to the dilates u_r, $0 < r < 1$, gives the desired conclusion. ■

Lemma 3. *If u is positive and harmonic in $B_n \setminus \{0\}$ and $u(x) \to 0$ as $|x| \to 1$, then there is a positive constant a such that*

$$u(x) = a(|x|^{2-n} - 1)$$

for all $x \in B_n \setminus \{0\}$.

Proof: By Lemma 1, we need only show that $u = A[u]$ in $B_n \setminus \{0\}$. Suppose we could show that $u \geqslant A[u]$ in $B_n \setminus \{0\}$. Then if there were a point $x \in B_n \setminus \{0\}$ such that $u(x) > A[u](x)$, we would have

$$A[u](x) > A[A[u]](x) = A[u](x),$$

a contradiction. Thus we need only prove that $u \geqslant A[u]$ in $B_n \setminus \{0\}$, which we now do.

Let α be the constant of Lemma 2. Then by Lemma 1, $u - \alpha A[u]$ is harmonic in $B_n \setminus \{0\}$. By Lemma 2, $u(x) - \alpha A[u](x) > 0$ if $0 < |x| \leqslant 1/2$, and clearly $u(x) - \alpha A[u](x) \to 0$ as $|x| \to 1$ by our hypothesis on u. The minimum principle for harmonic functions thus shows that $u - \alpha A[u] > 0$ in $B_n \setminus \{0\}$.

We wish to iterate this result. For this purpose, define

$$f(t) = \alpha + t(1 - \alpha), \qquad t \in [0, 1].$$

Suppose we know

$$w = u - tA[u] > 0 \quad \text{in } B_n \setminus \{0\} \tag{$*$}$$

for some $t \in [0,1]$. Since $w(x) \to 0$ as $|x| \to 1$, the above argument may be applied to w, yielding

$$w - \alpha A[w] = u - f(t)A[u] > 0 \quad \text{in } B_n \setminus \{0\}.$$

This process may be continued. Letting $f^{(m)}$ denote the m^{th} iterate of f, we see that $(*)$ implies

$$u - f^{(m)}(t)A[u] > 0 \quad \text{in } B_n \setminus \{0\}$$

for $m = 1, 2 \ldots$. But $f^{(m)}(t) \to 1$ as $m \to \infty$, for every $t \in [0,1]$, so that $(*)$ holding for some $t \in [0,1]$ implies $u - A[u] \geqslant 0$ in $B_n \setminus \{0\}$. Since $(*)$ obviously holds when $t = 0$, we have $u - A[u] \geqslant 0$ in $B_n \setminus \{0\}$, as desired. ■

PROOF OF BÔCHER'S THEOREM. We first assume that u is positive and harmonic on a neighborhood of $\overline{B}_n \setminus \{0\}$. For $x \in B_n \setminus \{0\}$, define

$$w(x) = u(x) - P[u|_S](x) + |x|^{2-n} - 1;$$

here $P[u|_S]$ denotes the Poisson integral of $u|_S$ (the unique harmonic function in B_n that extends continuously to $\overline{B}_n$ with boundary values $u|_S$). As $|x| \to 1$, we have $w(x) \to 0$, and as $|x| \to 0$, we have $w(x) \to +\infty$. By the minimum principle, w is positive and harmonic in $B_n \setminus \{0\}$. Lemma 3, applied to w, shows that $u(x) = a|x|^{2-n} + v(x)$ in $B_n \setminus \{0\}$ for some v harmonic in B_n and some constant a. To finish the proof of Bôcher's Theorem in this special case, note that a must be nonnegative, because otherwise $u(x) \to -\infty$ as $x \to 0$, which would violate the positivity of u.

For the general positive harmonic u in $B_n \setminus \{0\}$, we may apply the above result to $u_{1/2}$, so that

$$u(x/2) = a|x|^{2-n} + v(x) \quad \text{in } B_n \setminus \{0\}$$

for some v harmonic in B_n and some constant $a \geqslant 0$. This implies

$$u(x) = a2^{2-n}|x|^{2-n} + v(2x) \quad \text{in } (\tfrac{1}{2})B_n \setminus \{0\},$$

which shows that $u(x) - a2^{2-n}|x|^{2-n}$ extends harmonically to $(\frac{1}{2})B_n$, and hence to B_n. The proof of Bôcher's Theorem is complete. ■

POSITIVE HARMONIC FUNCTIONS ON $\mathbf{R}^n \setminus \{0\}$. We conclude this note by characterizing the positive harmonic functions on $\mathbf{R}^n \setminus \{0\}$. The proof uses Bôcher's Theorem and the well known result that a positive harmonic function on all of $\mathbf{R}^n$ is constant (see Note 1 below).

Corollary.
(i) *If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then u is constant.*
(ii) *If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$ $(n > 2)$, then there are nonnegative constants a and b such that*

$$u(x) = a|x|^{2-n} + b$$

for all $x \in \mathbf{R}^n \setminus \{0\}$.

Proof: (i). If u is positive and harmonic on $\mathbf{R}^2 \setminus \{0\}$, then the function $u(e^z)$ is positive and harmonic on $\mathbf{R}^2$ $(= \mathbf{C})$ and hence is constant. This proves u is constant.

(ii). If u is positive and harmonic on $\mathbf{R}^n \setminus \{0\}$, we may write

$$u(x) = a|x|^{2-n} + v(x)$$

in $B_n \setminus \{0\}$, as in (ii) of Bôcher's Theorem. The function v extends harmonically to all of $\mathbf{R}^n$ by setting $v(x) = u(x) - a|x|^{2-n}$ for $x \in \mathbf{R}^n \setminus B_n$. We may thus apply the minimum principle to v: For any fixed $x \in \mathbf{R}^n$ and every $r > |x|$ we have

$$v(x) \geqslant \min\{v(\zeta): |\zeta| = r\} > -a|r|^{2-n},$$

where the positivity of u gives the second inequality. Letting $r \to \infty$, we see that v is nonnegative and harmonic on $\mathbf{R}^n$ and hence is constant. This completes the proof. ■

Notes. 1. For the convenience of the reader, we sketch a simple proof (inspired by Nelson [**6**]) that a positive harmonic function v on $\mathbf{R}^n$ is constant; for the standard proof see [**3**], Theorem 1.19. Let $B(x, r)$ denote the open ball in $\mathbf{R}^n$ with center x and radius r. Fix $x \in \mathbf{R}^n$, $x \neq 0$, and let $r > |x|$. The volume version of the mean value property shows that $(v(x) - v(0))V(B(0, r))$ equals the difference of the integrals of v over $B(x, r)$ and $B(0, r)$. In this difference the integral of v over $B(x, r) \cap B(0, r)$ cancels, making $|(v(x) - v(0))V(B(0, r))|$ less than the integral of v over the symmetric difference of these balls (we have used the positivity of v here). This integral is less than the integral of v over $B(0, r + |x|) \setminus B(0, r - |x|)$, which we may compute exactly using the volume mean value property. It follows that

$$|\nu(x) - \nu(0)| < \frac{(r + |x|)^n - (r - |x|)^n}{r^n} \nu(0).$$

The last term tends to zero as $r \to \infty$, and thus $\nu(x) = \nu(0)$, proving that ν is constant.

2. Another proof of Bôcher's Theorem, again quite different from the classical proofs, will appear in [**1**].

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