The Berezin Transform on the Toeplitz Algebra

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11 June 1997

Abstract. This paper studies the boundary behavior of the Berezin transform on the C^*-algebra generated by the analytic Toeplitz operators on the Bergman space.

1 Introduction

Let dA denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. The *Bergman space* L^2_a is the Hilbert space consisting of the analytic functions on D that are also in $L^2(D, dA)$. For $z \in D$, the *Bergman reproducing kernel* is the function $K_z \in L^2_a$ such that

$$f(z) = \langle f, K_z \rangle$$

for every $f \in L^2_a$. The *normalized Bergman reproducing kernel* k_z is the function $K_z/\|K_z\|_2$. Here, as elsewhere in this paper, the norm $\| \ \|_2$ and the inner product $\langle \ , \ \rangle$ are taken in the space $L^2(D, dA)$. The set of bounded operators on L^2_a is denoted by $\mathcal{B}(L^2_a)$.

For $S \in \mathcal{B}(L^2_a)$, the *Berezin transform* of S is the function $\tilde{S}$ on D defined by

$$\tilde{S}(z) = \langle Sk_z, k_z \rangle.$$

Often the behavior of the Berezin transform of an operator provides important information about the operator.

For $u \in L^\infty(D, dA)$, the *Toeplitz operator* T_u with symbol u is the operator on L^2_a defined by $T_u f = P(uf)$; here P is the orthogonal projection from $L^2(D, dA)$ onto L^2_a. Note that if $g \in H^\infty$ (the set of bounded analytic functions on D), then T_g is just the operator of multiplication by g on L^2_a.

Both authors were partially supported by the National Science Foundation. The first author also thanks the Mathematical Sciences Research Institute (Berkeley), for its hospitality while part of this work was in progress.

The Berezin transform $\tilde{u}$ of a function $u \in L^\infty(D, dA)$ is defined to be the Berezin transform of the Toeplitz operator T_u. In other words, $\tilde{u} = \widetilde{T_u}$. Because $\langle T_u k_z, k_z \rangle = \langle P(uk_z), k_z \rangle = \langle uk_z, k_z \rangle$, we obtain the formula

$$\tilde{u}(z) = \int_D u(w)|k_z(w)|^2 \, dA(w). \tag{1.1}$$

The Berezin transform of a function in $L^\infty(D, dA)$ often plays the same important role in the theory of Bergman spaces as the harmonic extension of a function in $L^\infty(\partial D, d\theta)$ plays in the theory of Hardy spaces.

The *Toeplitz algebra* $\mathcal{T}$ is the C^*-subalgebra of $\mathcal{B}(L^2_a)$ generated by $\{T_g : g \in H^\infty\}$. We let $\mathcal{U}$ denote the C^*-subalgebra of $L^\infty(D, dA)$ generated by H^∞. As is well known (see [2], Proposition 4.5), $\mathcal{U}$ equals the closed subalgebra of $L^\infty(D, dA)$ generated by the set of bounded harmonic functions on D. Although the map $u \mapsto T_u$ is not multiplicative on $L^\infty(D, dA)$, the identities $T_u{}^* = T_{\bar{u}}$, $T_u T_g = T_{ug}$, and $T_{\bar{g}} T_u = T_{\bar{g}u}$ hold for all $u \in L^\infty$ and all $g \in H^\infty$. This implies that $\mathcal{T}$ equals the closed subalgebra of $\mathcal{B}(L^2_a)$ generated by the Toeplitz operators with bounded harmonic symbol, and that $\mathcal{T}$ also equals the closed subalgebra of $\mathcal{B}(L^2_a)$ generated by $\{T_u : u \in \mathcal{U}\}$. Our goal in this paper is to study the boundary behavior of the Berezin transforms of the operators in $\mathcal{T}$ and of the functions in $\mathcal{U}$.

In Section 2 we study the boundary behavior of Berezin transforms of operators in $\mathcal{T}$. We show (Theorem 2.11) that if $S \in \mathcal{T}$, then $\tilde{S} \in \mathcal{U}$. Perhaps the main result in this section is Theorem 2.16, which describes the commutator ideal $\mathcal{C}_\mathcal{T}$ (the smallest closed, two-sided ideal of $\mathcal{T}$ containing all operators of the form $RS - SR$, where $R, S \in \mathcal{T}$). As a consequence of this result, we show (Corollary 2.17) that $S - T_{\tilde{S}}$ is in the commutator ideal $\mathcal{C}_\mathcal{T}$ for every $S \in \mathcal{T}$. Writing $S = T_{\tilde{S}} + (S - T_{\tilde{S}})$, this gives us a canonical way to express the (nondirect) sum $\mathcal{T} = \{T_u : u \in \mathcal{U}\} + \mathcal{C}_\mathcal{T}$. We also prove (Corollary 2.19) that if $S \in \mathcal{C}_\mathcal{T}$, then $\tilde{S}$ has nontangential limit 0 at almost every point of ∂D.

In Section 3 we study the boundary behavior of Berezin transforms of functions in $\mathcal{U}$. We prove (Corollary 3.4) that if $u \in \mathcal{U}$, then $\tilde{u} - u$ has nontangential limit 0 at almost every point of ∂D. Using similar techniques, we prove (Corollary 3.7) that if $u \in \mathcal{U}$, then the function $z \mapsto \|T_{u-u(z)} k_z\|_2$ has nontangential limit 0 at almost every point of ∂D. The main result of this section is Theorem 3.10, which describes the functions $u \in \mathcal{U}$ such that $\tilde{u}(z) - u(z) \to 0$ as $z \to \partial D$. As a consequence, we describe (Corollary 3.12) the operators S that differ from the Toeplitz operator $T_{\tilde{S}}$ by a compact operator, where S is a finite sum of finite products of Toeplitz operators with symbols in $\mathcal{U}$.

In Section 4 we use results from the two previous sections to describe (Theorem 4.5) when the Berezin transform is asymptotically multiplicative on harmonic functions. This theorem is then used to characterize the functions $f, g \in H^\infty$ such that $T_{\bar{f}} T_g - T_g T_{\bar{f}}$ is compact.

We thank Jarosław Lech for useful conversations about the Berezin transform.

2 Boundary Behavior of the Berezin Transform on $\mathcal{T}$

In this section we will study the boundary behavior of the Berezin transform on elements of $\mathcal{T}$. We will need explicit formulas for the reproducing kernel and the normalized reproducing kernel. As is well known,

$$K_z(w) = \frac{1}{(1 - \bar{z}w)^2}$$

for $z, w \in D$. Note that

$$\|K_z\|_2{}^2 = \langle K_z, K_z \rangle = K_z(z) = \frac{1}{(1 - |z|^2)^2}.$$

Thus

$$k_z(w) = \frac{1 - |z|^2}{(1 - \bar{z}w)^2} \tag{2.1}$$

for $z, w \in D$.

Analytic automorphisms of the unit disk will play a key role here. For $z \in D$, let φ_z be the Möbius map on D defined by

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}. \tag{2.2}$$

Let $U_z \colon L^2_a \to L^2_a$ be the unitary operator defined by

$$U_z f = (f \circ \varphi_z)\varphi_z{}'. \tag{2.3}$$

To show that U_z is indeed unitary, first make a change of variables in the integral defining $\|U_z f\|_2$ to show that U_z is an isometry on L^2_a. Next, a simple computation shows that $U_z{}^2$ is the identity operator on L^2_a (this holds because φ_z is its own inverse under composition). Being an invertible isometry, U_z must be unitary. Notice that $U_z{}^* = U_z{}^{-1} = U_z$, so U_z is actually a self-adjoint unitary operator.

We will need two more simple properties of U_z. First,

$$U_z 1 = -k_z; \tag{2.4}$$

this follows from (2.1), (2.2), and (2.3). Second,

$$U_z T_u U_z = T_{u \circ \varphi_z} \tag{2.5}$$

for every $u \in L^\infty(D, dA)$; this is proved as Lemma 8 of [3]. Thus if $u_1, \ldots, u_n \in L^\infty(D, dA)$, then

$$U_z T_{u_1} \ldots T_{u_n} U_z = T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z} \tag{2.6}$$

because we can write the operator on the left side as

$$(U_z T_{u_1} U_z)(U_z T_{u_2} U_z) \ldots (U_z T_{u_n} U_z)$$

and then use (2.5).

Next we compute the Berezin transform of a product of Toeplitz operators. The formula given by the following lemma will be used later when we prove that $\tilde{S} \in \mathcal{U}$ for every $S \in \mathcal{T}$ (Theorem 2.11).

Lemma 2.7 *If $u_1, \ldots, u_n \in L^\infty(D, dA)$, then*

$$(T_{u_1} \ldots T_{u_n})\tilde{\ }(z) = \langle T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z} 1, 1 \rangle$$

for every $z \in D$.

Proof: Suppose $u_1, \ldots, u_n \in L^\infty(D, dA)$ and $z \in D$. Then

$$\begin{aligned} (T_{u_1} \ldots T_{u_n})\tilde{\ }(z) &= \langle T_{u_1} \ldots T_{u_n} k_z, k_z \rangle \\ &= \langle T_{u_1} \ldots T_{u_n} U_z 1, U_z 1 \rangle \\ &= \langle U_z T_{u_1} \ldots T_{u_n} U_z 1, 1 \rangle \\ &= \langle T_{u_1 \circ \varphi_z} \ldots T_{u_n \circ \varphi_z} 1, 1 \rangle, \end{aligned}$$

where the first equality comes from the definition of the Berezin transform, the second equality comes from (2.4), the third equality holds because U_z is self-adjoint, and the last equality comes from (2.6). ∎

We will need to make extensive use of the *maximal ideal space* of H^∞, which we denote by $\mathcal{M}$. We define $\mathcal{M}$ to be the set of multiplicative linear maps from H^∞ onto the field of complex numbers. With the weak-star topology, $\mathcal{M}$ is a compact Hausdorff space. If z is a point in the unit disk D, then point evaluation at z is a multiplicative linear functional on $\mathcal{M}$. Thus we can think of z as an element of $\mathcal{M}$ and the unit disk D as a subset of $\mathcal{M}$. Carleson's corona theorem states that D is dense in $\mathcal{M}$.

Suppose $m \in \mathcal{M}$ and $z \mapsto \alpha_z$ is a mapping of D into some topological space E. Suppose also that $\beta \in E$. The notation

$$\lim_{z \to m} \alpha_z = \beta$$

means (as you should expect) that for each open set X in E containing β, there is an open set Y in $\mathcal{M}$ containing m such that $\alpha_z \in X$ for all $z \in Y \cap D$. Note that with this notation z is always assumed to lie in D. We must deal with these nets rather than sequences because the topology of $\mathcal{M}$ is not metrizable.

The Gelfand transform allows us to think of H^∞ as contained in $C(\mathcal{M})$, the algebra of continuous complex-valued functions on $\mathcal{M}$. By the Stone-Weierstrass theorem, the set of finite sums of functions of the form $f\bar{g}$, with $f, g \in H^\infty$, is dense in $C(\mathcal{M})$, where $C(\mathcal{M})$ is endowed with the usual supremum norm. Because D is dense in $\mathcal{M}$, this supremum norm is the same as the usual supremum norm over D. Thus we can identify $C(\mathcal{M})$ with $\mathcal{U}$, the closure in $L^\infty(D, dA)$ of finite sums of functions of the form $f\bar{g}$, with $f, g \in H^\infty$.

We will make frequent use of the identification discussed above of $\mathcal{U}$ with $C(\mathcal{M})$. It asserts that given a function $u \in \mathcal{U}$, which we normally think of as a function on D, we can uniquely extend u to a continuous complex-valued function on $\mathcal{M}$; this extension to $\mathcal{M}$ is also denoted by u. Thus for $u \in \mathcal{U}$ and $m \in \mathcal{M}$, the expression $u(m)$ makes sense—it is the complex number defined by

$$u(m) = \lim_{z \to m} u(z).$$

Conversely, we will sometimes use the identification of $\mathcal{U}$ with $C(\mathcal{M})$ to prove that a function is in $\mathcal{U}$. Specifically, if u is a continuous function on D and we can prove that u extends to a continuous function on $\mathcal{M}$, then we can conclude that $u \in \mathcal{U}$.

For $m \in \mathcal{M}$, let $\varphi_m \colon D \to \mathcal{M}$ denote the *Hoffman map*. This is defined by setting

$$\varphi_m(w) = \lim_{z \to m} \varphi_z(w)$$

for $w \in D$; here we are taking a limit in $\mathcal{M}$. The existence of this limit, as well as many other deep properties of φ_m, was proved by Hoffman [9]. An exposition of Hoffman's results can also be found in [8], Chapter X. We shall use, without further comment, Hoffman's result that φ_m is a continuous mapping of D into $\mathcal{M}$. Note that $\varphi_m(0) = m$.

If $u \in \mathcal{U}$ and $m \in \mathcal{M}$, then $u \circ \varphi_m$ makes sense as a continuous function on D, because φ_m maps D into $\mathcal{M}$ and u can be thought of as a continuous function on $\mathcal{M}$, as we discussed above. The next lemma provides the crucial continuity that we will soon need. Recall that a net of operators $\{S_z\}_{z\in D} \subset \mathcal{B}(L_a^2)$ is said to converge to $S \in \mathcal{B}(L_a^2)$ in the *strong operator topology* as $z \to m$ if $\lim_{z\to m} S_z f = Sf$ for every $f \in L_a^2$, where the last limit is taken in the norm in L_a^2.

Lemma 2.8 *If* $u_1, \ldots, u_n \in \mathcal{U}$, *then*

$$\lim_{z\to m} T_{u_1\circ\varphi_z} \ldots T_{u_n\circ\varphi_z} = T_{u_1\circ\varphi_m} \ldots T_{u_n\circ\varphi_m} \tag{2.9}$$

for every $m \in \mathcal{M}$, *where the limit is taken in the strong operator topology.*

PROOF: Fix $m \in \mathcal{M}$. We will prove (2.9) by induction on n. To get the induction started, suppose $n = 1$, so we consider a single function $u \in \mathcal{U}$. As $z \to m$, clearly $u \circ \varphi_z$ converges to $u \circ \varphi_m$ pointwise on D. Because the family of functions $\{u \circ \varphi_z : z \in D\}$ is uniformly bounded, this convergence is uniform on each compact subset of D if u happens to be analytic on D. Thus the convergence is also uniform on each compact subset of D if u happens to be the product of an H^∞ function and the complex conjugate of an H^∞ function. Finite sums of such functions are dense in $\mathcal{U}$. Thus we can conclude that $u \circ \varphi_z$ converges to $u \circ \varphi_m$ (as $z \to m$) uniformly on each compact subset of D for arbitrary $u \in \mathcal{U}$. Fix $f \in L_a^2$. Then

$$\lim_{z\to m} \int_D |(u \circ \varphi_z)(w) - (u \circ \varphi_m)(w)|^2 |f(w)|^2 \, dA(w) = 0, \tag{2.10}$$

because D, and hence the integral above, can be broken into two pieces—a large compact subset of D (on which $u \circ \varphi_z$ converges uniformly to $u \circ \varphi_m$) and a set of small measure on which all the integrals are small. (We had to use uniform convergence on compact subsets of D to prove (2.10) because the Lebesgue dominated convergence theorem fails for nets, as opposed to sequences.) Because

$$\lim_{z\to m} \|(u \circ \varphi_z) f - (u \circ \varphi_m) f\|_2 = 0,$$

we have $\lim_{z\to m} \|T_{u\circ\varphi_z} f - T_{u\circ\varphi_m} f\|_2 = 0$, proving (2.9) in the case $n = 1$.

Now suppose that $u_1, \ldots, u_n \in \mathcal{U}$ and that (2.9) holds when n is replaced by $n-1$. For convenience, let

$$S_z = T_{u_1 \circ \varphi_z} \ldots T_{u_{n-1} \circ \varphi_z} \quad \text{and} \quad S_m = T_{u_1 \circ \varphi_m} \ldots T_{u_{n-1} \circ \varphi_m}.$$

By our induction hypothesis, $\|S_z g - S_m g\|_2 \to 0$ as $z \to m$ for every $g \in L^2_a$. Fix $f \in L^2_a$. Then

$$\begin{aligned}
\|T_{u_1 \circ \varphi_z} &\ldots T_{u_n \circ \varphi_z} f - T_{u_1 \circ \varphi_m} \ldots T_{u_n \circ \varphi_m} f\|_2 \\
&= \|S_z T_{u_n \circ \varphi_z} f - S_m T_{u_n \circ \varphi_m} f\|_2 \\
&\leq \|S_z\|_2 \|(T_{u_n \circ \varphi_z} - T_{u_n \circ \varphi_m}) f\|_2 + \|(S_z - S_m)(T_{u_n \circ \varphi_m} f)\|_2.
\end{aligned}$$

Because $\|S_z\|_2$ is bounded by $\|u_1\|_\infty \ldots \|u_n\|_\infty$, which is independent of z, the first term in the last inequality above has limit 0 as $z \to m$ (by the $n = 1$ case that we already proved). Our induction hypothesis implies that the second term in the last inequality above also has limit 0 as $z \to m$, completing the proof of (2.9). ■

Now we are ready to prove that the Berezin transform maps $\mathcal{T}$ into $\mathcal{U}$. Most of the work needed to prove the theorem below was done in the last two lemmas. For the first time we will need to use the linearity of the Berezin transform as well as its continuity: $\|\tilde{S}\|_\infty \leq \|S\|$ for all $S \in \mathcal{B}(L^2_a)$. We will also need to make use of the description of $\mathcal{T}$ as the closure in $\mathcal{B}(L^2_a)$ of the set of finite sums of operators of the form $T_{u_1} \ldots T_{u_n}$, where $u_1, \ldots, u_n \in \mathcal{U}$.

Theorem 2.11 *If $S \in \mathcal{T}$, then $\tilde{S} \in \mathcal{U}$. Furthermore, if $u_1, \ldots, u_n \in \mathcal{U}$, then*

$$(T_{u_1} \ldots T_{u_n})\tilde{}\,(m) = \langle T_{u_1 \circ \varphi_m} \ldots T_{u_n \circ \varphi_m} 1, 1 \rangle \tag{2.12}$$

for every $m \in \mathcal{M}$.

Proof: If $u_1, \ldots, u_n \in \mathcal{U}$, then Lemma 2.7 and Lemma 2.8 show that $(T_{u_1} \ldots T_{u_n})\tilde{}$ extends to be a continuous function on $\mathcal{M}$ and that the extension is given by (2.12). Thus the Berezin transform maps sums of operators of the form $T_{u_1} \ldots T_{u_n}$, where each $u_j \in \mathcal{U}$, into $\mathcal{U}$. The linearity and continuity of the Berezin transform now imply that the Berezin transform also maps $\mathcal{T}$ into $\mathcal{U}$. ■

A multiplicative linear function $m \in \mathcal{M}$ is called a *one-point part* if φ_m is a constant map. In other words, m is a one-point part if $\varphi_m(w) = m$

for every $w \in D$. The set of all one-point parts is denoted by $\mathcal{M}_1$. As is well known, $\mathcal{M}_1$ is a closed subset of $\mathcal{M}$ that properly contains the Shilov boundary of H^∞ (in particular, $\mathcal{M}_1$ is not the empty set). Actually $\mathcal{M}_1$ should be thought of as a small subset of $\mathcal{M} \setminus D$, as the complement of $\mathcal{M}_1$ in $\mathcal{M} \setminus D$ is dense in $\mathcal{M} \setminus D$.

The following corollary shows how to compute the Berezin transform on $\mathcal{M}_1$ of a finite product of Toeplitz operators with symbols in $\mathcal{U}$.

Corollary 2.13 *If* $u_1, \dots, u_n \in \mathcal{U}$, *then*

$$(T_{u_1} \dots T_{u_n})\tilde{}(m) = u_1(m) \dots u_n(m) \tag{2.14}$$

for every $m \in \mathcal{M}_1$.

PROOF: Suppose $u_1, \dots, u_n \in \mathcal{U}$ and $m \in \mathcal{M}_1$. Because $m \in \mathcal{M}_1$, each function $u_j \circ \varphi_m$ is a constant function equal to the constant $u_j(m)$. Thus each of the Toeplitz operators in (2.12) has constant symbol, reducing (2.12) to the desired equation (2.14). ∎

The next corollary shows that the Berezin transform is multiplicative on $\mathcal{M}_1$.

Corollary 2.15 *If* $R, S \in \mathcal{T}$, *then*

$$(RS)\tilde{}(m) = \tilde{R}(m)\tilde{S}(m)$$

for every $m \in \mathcal{M}_1$.

PROOF: If R, S are each products of Toeplitz operators with symbols in $\mathcal{U}$, then the desired result follows from Corollary 2.13. The proof is completed by recalling that sums of such operators are dense in $\mathcal{T}$. ∎

Recall that the *commutator ideal* $\mathcal{C}_\mathcal{T}$ is the smallest closed, two-sided ideal of $\mathcal{T}$ containing all operators of the form $RS - SR$, where $R, S \in \mathcal{T}$. In a remarkable theorem, McDonald and Sundberg ([11], Theorem 6; also see [13] for another proof) showed that $\mathcal{T}/\mathcal{C}_\mathcal{T}$ is isomorphic, as a C^*-algebra, to $C(\mathcal{M}_1)$. More precisely, they showed that the map $u \mapsto T_u + \mathcal{C}_\mathcal{T}$ is a surjective homomorphism of $\mathcal{U}$ onto $\mathcal{T}/\mathcal{C}_\mathcal{T}$, with kernel $\{u \in \mathcal{U} : u|_{\mathcal{M}_1} = 0\}$. Rephrased again, the McDonald-Sundberg theorem states that each $S \in \mathcal{T}$ can be written in the form $S = T_u + R$ for some $u \in \mathcal{U}$ and some $R \in \mathcal{C}_\mathcal{T}$. Furthermore, if $u \in \mathcal{U}$, then $T_u \in \mathcal{C}_\mathcal{T}$ if and only if $u|_{\mathcal{M}_1} = 0$. These results account for the importance of understanding the commutator ideal $\mathcal{C}_{\mathcal{T}}$. We now describe $\mathcal{C}_{\mathcal{T}}$ in terms of Berezin transforms.

Theorem 2.16 *Suppose $S \in \mathcal{T}$. Then S is in the commutator ideal $\mathcal{C}_{\mathcal{T}}$ if and only if $\tilde{S}|_{\mathcal{M}_1} = 0$.*

PROOF: The commutator ideal $\mathcal{C}_{\mathcal{T}}$ is the norm closure of the set of finite sums of operators of the form $S_1(S_2S_3 - S_3S_2)S_4$, where $S_1, S_2, S_3, S_4 \in \mathcal{T}$. By Corollary 2.15, each such operator has a Berezin transform that vanishes on $\mathcal{M}_1$. Thus if $S \in \mathcal{C}_{\mathcal{T}}$, then $\tilde{S}|_{\mathcal{M}_1} = 0$, proving one direction of the theorem.

To prove the other direction, suppose $\tilde{S}|_{\mathcal{M}_1} = 0$. By the McDonald-Sundberg theorem, we can write $S = T_u + R$ for some $u \in \mathcal{U}$ and $R \in \mathcal{C}_{\mathcal{T}}$. Thus

$$\begin{aligned} 0 &= \tilde{S}|_{\mathcal{M}_1} \\ &= \widetilde{T_u}|_{\mathcal{M}_1} + \tilde{R}|_{\mathcal{M}_1} \\ &= u|_{\mathcal{M}_1} + \tilde{R}|_{\mathcal{M}_1} \\ &= u|_{\mathcal{M}_1}, \end{aligned}$$

where the third equality comes from Corollary 2.13 and the fourth equality holds by the direction of this theorem that we have already proved. The McDonald-Sundberg theorem now tells us that $T_u \in \mathcal{C}_{\mathcal{T}}$ (because $u|_{\mathcal{M}_1} = 0$). Therefore $S \in \mathcal{C}_{\mathcal{T}}$, completing the proof. ■

Given an operator $S \in \mathcal{T}$, the McDonald-Sundberg theorem tells us that S can be written in the form $S = T_u + R$ for some $u \in \mathcal{U}$ and $R \in \mathcal{C}_{\mathcal{T}}$. The choice of u is not unique, as it can be perturbed by any function in $\mathcal{U}$ that vanishes on $\mathcal{M}_1$. However, we now show that there is a canonical choice of u, namely the Berezin transform of S. The corollary below states that the decomposition

$$S = T_{\tilde{S}} + (S - T_{\tilde{S}})$$

satisfies the requirements of the McDonald-Sundberg theorem, because the term in parentheses is in $\mathcal{C}_{\mathcal{T}}$.

Corollary 2.17 *If $S \in \mathcal{T}$, then $S - T_{\tilde{S}} \in \mathcal{C}_{\mathcal{T}}$.*

PROOF: Suppose $S \in \mathcal{T}$. Then by Theorem 2.11, $\tilde{S} \in \mathcal{U}$. If $m \in \mathcal{M}_1$, then using Corollary 2.13 (with $n = 1$) we get

$$(S - T_{\tilde{S}})\tilde{\ }(m) = \tilde{S}(m) - \tilde{S}(m) = 0.$$

In other words, $(S - T_{\tilde{S}})^{\sim}|_{\mathcal{M}_1} = 0$. Thus Theorem 2.16 implies that $S - T_{\tilde{S}} \in \mathcal{C}_{\mathcal{T}}$, completing the proof. ∎

The next lemma will allow us to translate results about $\mathcal{M}_1$, a rather abstract object, into results about nontangential behavior on the unit disk D. When we refer to "almost every point of ∂D", we mean with respect to the usual linear Lebesgue (arc length) measure on ∂D.

Lemma 2.18 *If $u \in \mathcal{U}$ and $u|_{\mathcal{M}_1} = 0$, then u has nontangential limit 0 at almost every point of ∂D.*

PROOF: As is well known, every function in H^∞ has a nontangential limit at almost every point of ∂D. Thus every finite sum of functions of the form $f\bar{g}$, where $f, g \in H^\infty$, has a nontangential limit at almost every point of ∂D. Hence any function on D that is the uniform limit of a sequence of such functions also has a nontangential limit at almost every point of ∂D (this holds because the union of a countable collection of sets of measure 0 has measure 0). In other words, every function in $\mathcal{U}$ has a nontangential limit at almost every point of ∂D.

Suppose $u \in \mathcal{U}$ and $u|_{\mathcal{M}_1} = 0$. Define a function u^* (almost everywhere) on ∂D by letting $u^*(\lambda)$ equal the nontangential limit of u at $\lambda \in \partial D$. Let $X \subset \mathcal{M}$ denote the Shilov boundary of H^∞. By Theorem 11 of Axler and Shields's paper [5], the essential range of u^* on ∂D equals $u(X)$. However, X is contained in $\mathcal{M}_1$, so we conclude that the essential range of u^* on ∂D is just $\{0\}$. Thus u^* equals 0 almost everywhere on ∂D. Hence u has nontangential limit 0 at almost every point of ∂D. ∎

Now we can prove that the Berezin transform of each operator in the commutator ideal of $\mathcal{T}$ has nontangential limit 0 almost everywhere on ∂D.

Corollary 2.19 *If $S \in \mathcal{C}_{\mathcal{T}}$, then $\tilde{S}$ has nontangential limit 0 at almost every point of ∂D.*

PROOF: Combine Theorem 2.16 and Lemma 2.18 to obtain the desired result. ∎

The converse of the corollary above is false. To see this, let u be a function in $C(\mathcal{M})$ that equals 0 on the Shilov boundary of H^∞ but that is not identically 0 on $\mathcal{M}_1$. Then $\widetilde{T_u}$ equals 0 on the Shilov boundary of H^∞ (by Corollary 2.13). The proof of Lemma 2.18 thus shows that $\widetilde{T_u}$ has nontangential limit 0 at almost every point of ∂D. However, $\widetilde{T_u}$ is not identically 0 on $\mathcal{M}_1$ (by Corollary 2.13) and thus T_u is not in $\mathcal{C}_\mathcal{T}$ (by Theorem 2.16), providing the desired example.

The next lemma will be used in the proof of Proposition 2.26. In equation (2.21) below, 1 denotes the constant function on D (the function that maps z to 1) and z denotes the identity function on D (the function that maps z to z).

Lemma 2.20 *If $S \in \mathcal{B}(L^2_a)$, then $\tilde{S}$ is real analytic on D and*

$$(\Delta\tilde{S})(0) = 16\langle Sz, z\rangle - 8\langle S1, 1\rangle. \tag{2.21}$$

PROOF: Let $S \in \mathcal{B}(L^2_a)$. Define a complex-valued function F on $D \times D$ by

$$F(w, z) = \langle SK_{\bar{w}}, K_z\rangle$$

for $w, z \in D$. Note that here we are using the unnormalized reproducing kernels. For fixed $w \in D$, the function $SK_{\bar{w}}$ is in L^2_a, and hence is analytic on D. Because $F(w, z) = (SK_{\bar{w}})(z)$, this implies that $F(w, z)$ is analytic in z for fixed w. Similarly, for fixed $z \in D$, the function S^*K_z is in L^2_a, and hence is analytic on D. Because $F(w, z) = \overline{(S^*K_z)(\bar{w})}$, this implies that $F(w, z)$ is analytic in w for fixed z. Because F is analytic in each variable separately, we conclude that F is holomorphic on $D \times D$. Clearly $\tilde{S}(z) = (1 - |z|^2)^2 F(\bar{z}, z)$. Because F is holomorphic on $D \times D$, this implies that $\tilde{S}$ is real analytic on D, as desired.

To prove (2.21), we first express the explicit formula (2.1) for the normalized reproducing kernel as a power series:

$$k_z(w) = (1 - |z|^2)\sum_{j=0}^{\infty}(j+1)\bar{z}^j w^j.$$

Thus

$$\begin{aligned}
\tilde{S}(z) &= \langle Sk_z, k_z\rangle \\
&= (1 - |z|^2)^2 \sum_{j,n=0}^{\infty} (j+1)(n+1)\langle Sw^j, w^n\rangle \bar{z}^j z^n && (2.22)\\
&= (1 - 2z\bar{z} + z^2\bar{z}^2) \sum_{j,n=0}^{\infty} (j+1)(n+1)\langle Sw^j, w^n\rangle \bar{z}^j z^n && (2.23)\\
&= \sum_{j,n=0}^{\infty} a_{j,n}\bar{z}^j z^n, && (2.24)
\end{aligned}$$

where the coefficients $a_{j,n}$ could be computed explicitly. Note that

$$(\Delta\tilde{S})(0) = 4\frac{\partial^2 \tilde{S}}{\partial\bar{z}\,\partial z}(0)$$
$$= 4a_{1,1},$$

where the last equation follows from (2.24). From (2.23) we see that

$$a_{1,1} = 4\langle Sw, w\rangle - 2\langle S1, 1\rangle.$$

The proof of (2.21) is completed by combining the last two equations and replacing the independent variable w above (denoting the identity function) with the more common symbol z. ∎

We note for later use that (2.22) implies that an operator $S \in \mathcal{B}(L^2_a)$ is uniquely determined by its Berezin transform. To see this, suppose $S \in \mathcal{B}(L^2_a)$ and $\tilde{S}$ is identically 0 on D. We need to show that $S = 0$. Differentiating the infinite sum in (2.22) n times with respect to z and j times with respect to $\bar{z}$ and then evaluating at $z = 0$ shows that $\langle Sw^j, w^n\rangle = 0$ for all nonnegative integers j and n. Because finite linear combinations of $\{w^j : j \geq 0\}$ are dense in L^2_a, this implies that $S = 0$, as desired.

For $S \in \mathcal{B}(L^2_a)$ and $z \in D$, define $S_z \in \mathcal{B}(L^2_a)$ by

$$S_z = U_z S U_z.$$

Recall that U_z was defined by equation (2.3). Note that if S is a finite product of Toeplitz operators, then a formula for S_z is given by (2.6). The next lemma shows us how to define S_m for each $m \in \mathcal{M}$ in a manner consistent with the definition just given for S_z. This operator S_m plays an important role in Proposition 2.26, where it is used in the proofs of parts (b) and (c) even though it is not explicitly mentioned in the statements of those results.

Lemma 2.25 *If $S \in \mathcal{T}$ and $m \in \mathcal{M}$, then there exists $S_m \in \mathcal{T}$ such that*

$$\lim_{z \to m} S_z = S_m,$$

where the limit is taken in the strong operator topology. If $S = T_{u_1} \dots T_{u_n}$, where $u_1, \dots, u_n \in \mathcal{U}$, then $S_m = T_{u_1 \circ \varphi_m} \dots T_{u_n \circ \varphi_m}$.

PROOF: Fix $m \in \mathcal{M}$. First suppose $S = T_{u_1} \dots T_{u_n}$, where $u_1, \dots, u_n \in \mathcal{U}$. Then $S_z = T_{u_1 \circ \varphi_z} \dots T_{u_n \circ \varphi_z}$ for every $z \in D$, as we saw in (2.6). Thus, by Lemma 2.8,

$$\lim_{z \to m} S_z = T_{u_1 \circ \varphi_m} \dots T_{u_n \circ \varphi_m}.$$

To prove that the operator on the right side of this equation is in $\mathcal{T}$, we must show that $u \circ \varphi_m \in \mathcal{U}$ whenever $u \in \mathcal{U}$. Clearly this holds if $u \in H^\infty$, because then $u \circ \varphi_m \in H^\infty$. Taking complex conjugates and then products, we have that $u \circ \varphi_m \in \mathcal{U}$ for all u of the form $f\bar{g}$, where $f, g \in H^\infty$. Finite sums of such functions are dense in $\mathcal{U}$, showing that $u \circ \varphi_m \in \mathcal{U}$ for all $u \in \mathcal{U}$, as desired. This completes the proof of the lemma when S has the special form $T_{u_1} \dots T_{u_n}$, where $u_1, \dots, u_n \in \mathcal{U}$.

Now suppose $S \in \mathcal{T}$. Fix $f \in L^2_a$. We must prove that $\lim_{z \to m} S_z f$ exists in L^2_a. To do this, suppose $\epsilon > 0$. Then there is an operator R that is a finite sum of operators of the form $T_{u_1} \dots T_{u_n}$, where each $u_j \in \mathcal{U}$, such that $\|S - R\| \le \epsilon$. Thus $\|S_z f - R_z f\|_2 \le \epsilon \|f\|_2$. From the paragraph above, we know that $R_z f$ converges (as $z \to m$) to a function $R_m f$. Thus

$$\limsup_{z \to m} \|S_z f - R_m f\|_2 \le \epsilon \|f\|_2.$$

Thus

$$\limsup_{z, w \to m} \|S_z f - S_w f\|_2 \le 2\epsilon \|f\|_2.$$

Because ϵ is an arbitrary positive number, this means that $S_z f$ is a Cauchy net in L^2_a (as $z \to m$). However, L^2_a is complete, and so this Cauchy net must converge, as desired.

From the first paragraph of this proof, we know that $S_m \in \mathcal{T}$ for all S in a dense subset of $\mathcal{T}$. The mapping $S \mapsto S_m$ is continuous (in the operator norm), so S_m must be in $\mathcal{T}$ for all $S \in \mathcal{T}$, completing the proof. ∎

A function $u \in \mathcal{U}$ is said to be *real analytic* on $\mathcal{M}$ if $u \circ \varphi_m$ is real analytic on D for every $m \in \mathcal{M}$. The next proposition tells us that the Berezin transform of any operator $S \in \mathcal{T}$ is real analytic on $\mathcal{M}$. Furthermore, for $m \in \mathcal{M}$ we get a formula for computing the Laplacian at 0 of $\tilde{S} \circ \varphi_m$. These results will be used in the next section of this paper.

Proposition 2.26 *Suppose $S \in \mathcal{T}$. Then*

(a) $\tilde{S} \circ \varphi_m = \widetilde{S_m}$ *for every $m \in \mathcal{M}$;*

(b) $\tilde{S}$ *is real analytic on $\mathcal{M}$;*

(c) $\big(\Delta(\tilde{S} \circ \varphi_m)\big)(w) = \lim_{z \to m} \dfrac{(1 - |\varphi_z(w)|^2)^2 (\Delta \tilde{S})(\varphi_z(w))}{(1 - |w|^2)^2}$
for every $w \in D, m \in \mathcal{M}$.

Proof: We begin by deriving a useful formula. Suppose $w, z \in D$. If

$f \in L^2_a$, then

$$\begin{aligned}\langle f, U_z K_w \rangle &= \langle U_z f, K_w \rangle \\ &= (U_z f)(w) \\ &= (f \circ \varphi_z)(w) {\varphi_z}'(w) \\ &= \langle f, \overline{{\varphi_z}'(w)} K_{\varphi_z(w)} \rangle.\end{aligned}$$

Thus $U_z K_w = \overline{{\varphi_z}'(w)} K_{\varphi_z(w)}$. Rewriting this in terms of the normalized reproducing kernels, we have

$$U_z k_w = \alpha k_{\varphi_z(w)} \tag{2.27}$$

for some complex constant α. Without doing a computation, we know that $|\alpha| = 1$, because $\|k_w\|_2 = \|k_{\varphi_z(w)}\|_2 = 1$ and U_z is unitary.

For the rest of the proof, fix $m \in \mathcal{M}$. To prove (a), fix $w \in D$. If $z \in D$, then

$$\begin{aligned}\tilde{S}\big(\varphi_z(w)\big) &= \langle S k_{\varphi_z(w)}, k_{\varphi_z(w)} \rangle \\ &= \langle S U_z k_w, U_z k_w \rangle \\ &= \langle U_z S U_z k_w, k_w \rangle,\end{aligned}$$

where the second equality comes from (2.27) along with the extra information that $|\alpha| = 1$. Taking limits of the first and last terms above as $z \to m$, we get $\tilde{S}\big(\varphi_m(w)\big) = \langle S_m k_w, k_w \rangle$. Thus (a) holds.

To prove (b), recall that $S_m \in \mathcal{T}$ (by Lemma 2.25). Thus $\widetilde{S_m}$ is real analytic on D (by Lemma 2.20). Now (a) shows that $\tilde{S} \circ \varphi_m$ is real analytic on D. We thus conclude that $\tilde{S}$ is real analytic on $\mathcal{M}$, completing the proof of (b).

To prove (c), fix $w \in D$. Then

$$\begin{aligned}(\Delta \tilde{S})(w) &= \frac{\big(\Delta(\tilde{S} \circ \varphi_w)\big)(0)}{(1 - |w|^2)^2} \\ &= \frac{(\Delta \widetilde{S_w})(0)}{(1 - |w|^2)^2} \\ &= \frac{16 \langle S U_w z, U_w z \rangle - 8 \langle S U_w 1, U_w 1 \rangle}{(1 - |w|^2)^2},\end{aligned}$$

where the first equality follows from a standard calculation, the second equality comes from (a), and the third inequality comes from Lemma 2.20. The equation above shows that the map $S \mapsto (\Delta\tilde{S})(w)$ is a continuous linear functional on $\mathcal{B}(L_a^2)$ with respect to the strong operator topology on $\mathcal{B}(L_a^2)$. Now fix $m \in \mathcal{M}$. Then

$$\begin{aligned}
\big(\Delta(\tilde{S} \circ \varphi_m)\big)(w) &= (\Delta\widetilde{S_m})(w) \\
&= \lim_{z \to m} (\Delta\widetilde{S_z})(w) \\
&= \lim_{z \to m} \big(\Delta(\tilde{S} \circ \varphi_z)\big)(w) \\
&= \lim_{z \to m} \frac{(1-|\varphi_z(w)|^2)^2(\Delta\tilde{S})(\varphi_z(w))}{(1-|w|^2)^2},
\end{aligned}$$

where the first equality holds by (a), the second equality holds by the continuity discussed earlier in this paragraph and Lemma 2.25, the third inequality holds by (a), and the fourth inequality holds by a standard calculation. This completes the proof of (c). ∎

3 Boundary Behavior of the Berezin Transform on $\mathcal{U}$

In this section we will study the boundary behavior of the Berezin transform on elements of $\mathcal{U}$. Recall that if $u \in L^\infty(D, dA)$, then the Berezin transform $\tilde{u}$ is the function on D defined by $\tilde{u} = \widetilde{T_u}$. This definition leads to the explicit formula (1.1).

Our next result states that if u is in $\mathcal{U}$, then so is $\tilde{u}$. If $u \in \mathcal{U}$, then $u \circ \varphi_m$ is bounded and continuous on D, so the integral appearing in the proposition below makes sense.

Proposition 3.1 *The Berezin transform maps $\mathcal{U}$ into $\mathcal{U}$. Furthermore, if $u \in \mathcal{U}$, then*

$$\tilde{u}(m) = \int_D (u \circ \varphi_m)(w)\, dA(w) \tag{3.2}$$

for every $m \in M$.

Proof: Suppose $u \in \mathcal{U}$. By definition, $\tilde{u} = \widetilde{T_u}$. Theorem 2.11 thus tells us that $\tilde{u} \in \mathcal{U}$. Furthermore, from (2.12), which is used in the second equality below, we have

$$\begin{aligned}\tilde{u}(m) &= \widetilde{T_u}(m)\\ &= \langle T_{u\circ\varphi_m}1, 1\rangle\\ &= \langle P(u\circ\varphi_m), 1\rangle\\ &= \langle u\circ\varphi_m, 1\rangle\\ &= \int_D (u\circ\varphi_m)(w)\,dA(w),\end{aligned}$$

for every $m \in M$, as desired. ■

If u is a bounded harmonic function on D, then so is $u\circ\varphi_z$ for each $z \in D$. The mean value property and (3.2) then imply that $\tilde{u}(z) = (u\circ\varphi_z)(0) = u(z)$ for each $z \in D$. In other words, every harmonic function equals its Berezin transform.

The next corollary shows that a function in $\mathcal{U}$ and its Berezin transform agree on the set of one-point parts.

Corollary 3.3 *If $u \in \mathcal{U}$, then $\tilde{u}|_{\mathcal{M}_1} = u|_{\mathcal{M}_1}$.*

PROOF: Suppose $u \in \mathcal{U}$ and $m \in \mathcal{M}_1$. Then $(u\circ\varphi_m)(w) = u(m)$ for every $w \in D$ (recall that $m \in \mathcal{M}_1$ implies that φ_m is a constant map on D). Thus (3.2) shows that $\tilde{u}(m) = u(m)$, as desired. ■

The next corollary shows that a function in $\mathcal{U}$ and its Berezin transform have the same nontangential limits almost everywhere on ∂D (recall from the proof of Lemma 2.18 that every function in $\mathcal{U}$ has nontangential limits almost everywhere on ∂D).

Corollary 3.4 *If $u \in \mathcal{U}$, then $\tilde{u}-u$ has nontangential limit 0 at almost every point of ∂D.*

PROOF: Suppose $u \in \mathcal{U}$. Then $\tilde{u} - u \in \mathcal{U}$ (from Proposition 3.1) and $(\tilde{u} - u)|_{\mathcal{M}_1} = 0$ (from Corollary 3.3). Lemma 2.18 now gives the desired result. ■

For $u \in L^\infty(D, dA)$, the *Hankel operator* with symbol u is the operator H_u from L^2_a to $L^2(D, dA) \ominus L^2_a$ defined by $H_u f = (1 - P)(uf)$. The next corollary shows that Toeplitz and Hankel operators with symbol in $\mathcal{U}$ behave nicely on normalized reproducing kernels corresponding to a net of points converging to a one-point part.

Corollary 3.5 *If $u \in \mathcal{U}$, then*

$$\lim_{z \to m} \|T_{u-u(m)} k_z\|_2 = 0$$

and

$$\lim_{z \to m} \|H_u k_z\|_2 = 0$$

for every $m \in \mathcal{M}_1$.

PROOF: Suppose $u \in \mathcal{U}$ and $m \in \mathcal{M}_1$. We claim that

$$\lim_{z \to m} \|(u - u(m))k_z\|_2 = 0. \tag{3.6}$$

Once this is proved, the proof will be done, because

$$\|T_{u-u(m)} k_z\|_2 = \|P\big((u - u(m))k_z\big)\|_2 \le \|(u - u(m))k_z\|_2$$

and

$$\|H_u k_z\|_2 = \|H_{u-u(m)} k_z\|_2 = \|(1 - P)\big((u - u(m))k_z\big)\|_2 \le \|(u - u(m))k_z\|_2.$$

To prove (3.6), note that

$$\begin{aligned} \lim_{z \to m} \|(u - u(m))k_z\|_2{}^2 &= \lim_{z \to m} \int_D |u(w) - u(m)|^2 |k_z(w)|^2 \, dA(w) \\ &= \lim_{z \to m} (|u - u(m)|^2)\tilde{}(z) \\ &= (|u - u(m)|^2)\tilde{}(m), \end{aligned}$$

where the last equality holds because $(|u - u(m)|^2)\tilde{}$ can be thought of as a continuous function on $\mathcal{M}$ (by Proposition 3.1). The function $|u - u(m)|^2$ is in $\mathcal{U}$, so it equals its Berezin transform on $\mathcal{M}_1$ (by Corollary 3.3). In particular, because $m \in \mathcal{M}_1$ and $|u - u(m)|^2$ equals 0 at m, the last quantity above equals 0, completing the proof. ∎

In the next corollary we once again translate a statement involving $\mathcal{M}_1$ into a more concrete statement.

Corollary 3.7 *If $u \in \mathcal{U}$, then the functions*

$$z \mapsto \|T_{u-u(z)} k_z\|_2 \quad \text{and} \quad z \mapsto \|H_u k_z\|_2$$

have nontangential limits 0 at almost every point of ∂D.

PROOF: Suppose $u \in \mathcal{U}$. As in the proof of the previous corollary, we need only show that $\|(u - u(z))k_z\|_2$ has nontangential limit 0 at almost every point of ∂D. To do this, note that

$$\begin{aligned}\|(u-u(z))k_z\|_2{}^2 &= \int_D |u(w)-u(z)|^2|k_z(w)|^2\,dA(w)\\ &= \int_D \big(|u(w)|^2 - 2\operatorname{Re}(\overline{u(z)}u(w)) + |u(z)|^2\big)|k_z(w)|^2\,dA(w)\\ &= \widetilde{|u|^2}(z) - 2\operatorname{Re}(\overline{u(z)}\tilde{u}(z)) + |u(z)|^2\end{aligned}$$

for $z \in D$. The equation above, along with Proposition 3.1, shows that the function $z \mapsto \|(u-u(z))k_z\|_2{}^2$ is in $\mathcal{U}$. The equation above, along with Corollary 3.3, shows that the function $z \mapsto \|(u-u(z))k_z\|_2{}^2$ is 0 on $\mathcal{M}_1$. Lemma 2.18 now gives the desired result. ∎

The next two lemmas will be useful in proving Theorem 3.10, which is the main result of this section. The formula for $(\Delta\tilde{u})(0)$ given by the first lemma below could be proved by differentiating twice under the integral in the explicit formula for $\tilde{u}$ obtained from (1.1) and (2.1). However we have avoided that computation in our proof by using the formula given by Lemma 2.20.

Lemma 3.8 *If $u \in L^\infty(D, dA)$, then $\tilde{u}$ is real analytic on D and*

$$(\Delta\tilde{u})(0) = 8\int_D u(z)(2|z|^2 - 1)\,dA(z).$$

Proof: Suppose $u \in L^\infty(D, dA)$. Then $\tilde{u} = \widetilde{T_u}$, and hence Lemma 2.20 implies that $\tilde{u}$ is real analytic on D. From Lemma 2.20 we also have

$$\begin{aligned}(\Delta\tilde{u})(0) &= (\Delta\widetilde{T_u})(0)\\ &= 16\langle T_u z, z\rangle - 8\langle T_u 1, 1\rangle\\ &= 16\langle uz, z\rangle - 8\langle u, 1\rangle\\ &= 8\int_D u(z)(2|z|^2-1)\,dA(z),\end{aligned}$$

completing the proof. ∎

The next lemma provides information about the Berezin transforms of functions analogous to the information about the Berezin transforms of operators provided by Proposition 2.26.

Lemma 3.9 *Suppose $u \in \mathcal{U}$. Then*

(a) $\tilde{u} \circ \varphi_m = (u \circ \varphi_m)\tilde{}$ *for every* $m \in \mathcal{M}$;

(b) $\tilde{u}$ *is real analytic on* $\mathcal{M}$;

(c) $\big(\Delta(\tilde{u} \circ \varphi_m)\big)(w) = \lim_{z \to m} \dfrac{(1 - |\varphi_z(w)|^2)^2 (\Delta \tilde{u})(\varphi_z(w))}{(1 - |w|^2)^2}$
for every $w \in D, m \in \mathcal{M}$.

Proof: Suppose $m \in \mathcal{M}$. Then

$$\begin{aligned} \tilde{u} \circ \varphi_m &= \widetilde{T_u} \circ \varphi_m \\ &= \big((T_u)_m\big)\tilde{} \\ &= \big(T_{u \circ \varphi_m}\big)\tilde{} \\ &= (u \circ \varphi_m)\tilde{}, \end{aligned}$$

where the second equality comes from Proposition 2.26(a) and the third equality comes from the second statement in Lemma 2.25. The equation above shows that (a) holds.

Because $\tilde{u} = \widetilde{T_u}$, (b) and (c) follow immediately from parts (b) and (c) of Proposition 2.26. ∎

Now we turn to the question of describing the functions $u \in \mathcal{U}$ such that $\lim_{z \to \partial D} \tilde{u}(z) - u(z) = 0$. Because the disk D is dense in $\mathcal{M}$, this is easily seen to be equivalent to the question of describing the functions $u \in \mathcal{U}$ such that $\tilde{u}$ equals u on $\mathcal{M} \setminus D$. We have seen that $\tilde{u}$ equals u on $\mathcal{M}_1$ for every $u \in \mathcal{U}$ (Corollary 3.3); now we are asking when equality holds on the larger set $\mathcal{M} \setminus D$. As motivation for our answer, recall that we pointed out earlier that every bounded harmonic function equals its Berezin transform. The converse also holds, so a function in $L^\infty(D, dA)$ equals its Berezin transform if and only if it is harmonic (for proofs of this deep result, see the papers by Engliš [7] or Ahern, Flores, and Rudin [1]). Thus we might guess that a function $u \in \mathcal{U}$ equals $\tilde{u}$ on $\mathcal{M} \setminus D$ if and only if u is harmonic on $\mathcal{M} \setminus D$ (whatever that means). As we will see, this turns out to be correct if we define the notion of harmonic on $\mathcal{M} \setminus D$ in terms of the parameterizations given by the Hoffman maps.

Motivated by the paragraph above, we define *HOP* (which stands for "harmonic on parts") to be the set of functions $u \in \mathcal{U}$ such that $u \circ \varphi_m$ is harmonic on D for every $m \in \mathcal{M} \setminus D$. Every bounded harmonic function on D is in *HOP*. (Proof: If u is a bounded harmonic function on D, then so is $u \circ \varphi_z$ for every $z \in D$. Now $u \circ \varphi_m(w) = \lim_{z \to m} u \circ \varphi_z(w)$ for every $w \in D, m \in \mathcal{M}$. Because the pointwise limit of any uniformly bounded net of harmonic functions is harmonic, we conclude that $u \circ \varphi_m$ is harmonic, as desired.) Every function in $C(\bar{D})$ is also in *HOP* (because if $u \in C(\bar{D})$ and $m \in \mathcal{M} \setminus D$, then $u \circ \varphi_m$ is a constant function on D).

The next theorem gives several conditions on a function $u \in \mathcal{U}$ that are equivalent to having $\lim_{z \to \partial D} \tilde{u}(z) - u(z) = 0$. Note that condition (h) in the theorem below would not make sense for an arbitrary $u \in \mathcal{U}$ (because functions in $\mathcal{U}$ need not even be differentiable on D). Even for a function $u \in \mathcal{U}$ that is differentiable on D, there is no obvious connection between the derivatives of u on D and derivatives of the functions $u \circ \varphi_m$ for $m \in \mathcal{M} \setminus D$. This helps explain the extra hypothesis required below for the applicability of condition (h).

Theorem 3.10 *Suppose $u \in \mathcal{U}$. Then the following are equivalent:*

(a) $\displaystyle\lim_{z \to \partial D} \tilde{u}(z) - u(z) = 0$;

(b) $\tilde{u} = u$ *on* $\mathcal{M} \setminus D$;

(c) $u \in$ *HOP*;

(d) $\tilde{u} \in$ *HOP*;

(e) $T_{\tilde{u}} - T_u$ *is a compact operator;*

(f) $\displaystyle\lim_{z \to \partial D} \int_D (u \circ \varphi_z)(w)(2|w|^2 - 1)\, dA(w) = 0$;

(g) $\displaystyle\lim_{z \to \partial D} (1 - |z|^2)^2 (\Delta \tilde{u})(z) = 0.$

If u is a finite sum of functions of the form $u_1 \dots u_n$, where each u_j is a bounded harmonic function on D, then the conditions above are also equivalent to the condition below:

(h) $\displaystyle\lim_{z \to \partial D} (1 - |z|^2)^2 (\Delta u)(z) = 0.$

Proof: As is well known, the equivalence of (a) and (b) follows from the corona theorem.

Suppose (b) holds, so $\tilde{u} = u$ on $\mathcal{M} \setminus D$. Let $m \in \mathcal{M} \setminus D$. Then

$$(u \circ \varphi_m)\tilde{} = \tilde{u} \circ \varphi_m = u \circ \varphi_m,$$

where the first equality comes from Lemma 3.9(a) and the second equality comes from our hypothesis (b). The equation above says that $u \circ \varphi_m$ is a function in $L^\infty(D, dA)$ that equals its Berezin transform. As we discussed earlier, Engliš [7] and Ahern, Flores, and Rudin [1] proved that only harmonic functions equal their Berezin transforms. Thus $u \circ \varphi_m$ is harmonic. Because m was an arbitrary element of $\mathcal{M} \setminus D$, this implies that $u \in HOP$. Thus (b) implies (c).

Now suppose (c) holds, so $u \in HOP$. If $m \in \mathcal{M} \setminus D$, then

$$\tilde{u} \circ \varphi_m = (u \circ \varphi_m)\tilde{} = u \circ \varphi_m,$$

where the first equality comes from Lemma 3.9(a) and the second equality holds because $u \circ \varphi_m$ is harmonic. The equation above shows that $\tilde{u} \circ \varphi_m$ is harmonic for all $m \in \mathcal{M} \setminus D$, which means that $\tilde{u} \in HOP$. Thus (c) implies (d).

Now suppose (d) holds, so $\tilde{u} \in HOP$. Let $m \in \mathcal{M} \setminus D$. Thus $\tilde{u} \circ \varphi_m$ is a harmonic function and hence is equal to its Berezin transform. In other words,

$$(\tilde{u} \circ \varphi_m)\tilde{} = \tilde{u} \circ \varphi_m = (u \circ \varphi_m)\tilde{},$$

where the second equality comes from Lemma 3.9(a). Because the Berezin transform is one-to-one (as we showed after the proof of Lemma 2.20), the equation above implies that

$$\tilde{u} \circ \varphi_m = u \circ \varphi_m.$$

Evaluating both sides of this equation at 0 shows that $\tilde{u}(m) = u(m)$. Thus (d) implies (b). At this point in the proof we have shown that (a), (b), (c), and (d) are equivalent.

A result of McDonald and Sundberg ([11], Proposition 5) states that for a function $v \in \mathcal{U}$, the Toeplitz operator T_v is compact if and only if $\lim_{z \to \partial D} v(z) = 0$. Applying this result with $v = \tilde{u} - u$ shows that (a) and (e) are equivalent. Thus we now know that (a) through (e) are equivalent.

By Lemma 3.8, eight times the integral in (f) equals $\big(\Delta(u \circ \varphi_z)\tilde{}\big)(0)$, which by Lemma 3.9(a) equals $\big(\Delta(\tilde{u} \circ \varphi_z)\big)(0)$, which by a standard calculation equals $(1 - |z|^2)^2 (\Delta \tilde{u})(z)$. Thus (f) and (g) are equivalent.

Now suppose that (d) holds, so $\tilde{u} \in HOP$. Thus

$$\big(\Delta(\tilde{u} \circ \varphi_m)\big)(0) = 0$$

for every $m \in \mathcal{M} \setminus D$. By Lemma 3.9(c) (with $w = 0$), this gives (g). Thus (d) implies (g).

Now suppose (g) holds. Fix $w \in D$ and $m \in \mathcal{M} \setminus D$. Note that $|\varphi_z(w)| \to 1$ as $z \to m$ (recall that w is fixed). From Lemma 3.9(c) and our hypothesis (g) we now conclude that $\big(\Delta(\tilde{u} \circ \varphi_m)\big)(w) = 0$. Thus $\tilde{u} \circ \varphi_m$ is harmonic on D. Because m was an arbitrary element of $\mathcal{M} \setminus D$, this means that $\tilde{u} \in HOP$. Thus (g) implies (d), completing the proof that (a) through (g) are equivalent.

To deal with (h), now suppose that u is a finite sum of functions of the form $u_1 \dots u_n$, where each u_j is a bounded harmonic function on D. For each such u_j, the function $u_j \circ \varphi_z$ is harmonic on D for every $z \in D$. If $m \in \mathcal{M}$, then $u_j \circ \varphi_z$ converges pointwise on D to $u_j \circ \varphi_m$ as $z \to m$. A pointwise convergent net of uniformly bounded harmonic functions has the property that every partial derivative (of arbitrary order) also converges pointwise to the appropriate partial derivative of the limit function. Applying this (and the appropriate product rule for partial derivatives) to u gives

$$\begin{aligned}\big(\Delta(u \circ \varphi_m)\big)(w) &= \lim_{z \to m} \big(\Delta(u \circ \varphi_z)\big)(w) \\ &= \lim_{z \to m} \frac{(1 - |\varphi_z(w)|^2)^2 (\Delta u)(\varphi_z(w))}{(1 - |w|^2)^2},\end{aligned}$$

for every $w \in D, m \in \mathcal{M}$, where the second equality comes from a standard calculation. To prove that (c) is equivalent to (h), now follow the pattern of the proof showing that (d) is equivalent to (g), using the last equality above in place of Lemma 3.9(c). ∎

A continuous bounded function on D can have Berezin transform in $C(\bar{D})$ without itself being in $C(\bar{D})$ (of course, to say that a continuous function on D is in $C(\bar{D})$ means that it extends continuously to a function on $\bar{D}$). To construct an example, consider a continuous function v on $[0, 1)$ that equals 0 most of the time (enough so that the average value of v on the interval $[r, 1)$ tends to 0 as r increases to 1), but whose graph occasionally has a small bump with height 1 (so that v does not extend continuously to $[0, 1]$). Define a radial function u on D by $u(z) = v(|z|)$. Then $\tilde{u}$ extends continuously to $\bar{D}$ even though u does not have this property. The following corollary shows that functions in $\mathcal{U}$ cannot behave in this fashion.

Corollary 3.11 *Suppose $u \in \mathcal{U}$. If $\tilde{u} \in C(\bar{D})$, then $u \in C(\bar{D})$.*

Proof: Suppose $\tilde{u} \in C(\bar{D})$. Then $\tilde{u} \in HOP$ (because $u \circ \varphi_m$ is constant on D for every $m \in \mathcal{M} \setminus D$). Because condition (d) in Theorem 3.10 holds, condition (a) in the same theorem also holds. Condition (a) and the continuity of $\tilde{u}$ on $\bar{D}$ imply that u extends continuously to $\bar{D}$, completing the proof. ■

Suppose $u \in \mathcal{U}$. In proving Theorem 3.10, we used the McDonald-Sundberg theorem that T_u is compact if and only if $u(z) \to 0$ as $z \to \partial D$ ([11], Proposition 5). To provide an easy proof of this theorem using our tools, note that Theorem 2.2 of our paper [6] asserts that T_u is compact if and only if $\tilde{u} \to 0$ as $z \to \partial D$. The equivalence of (a) and (d) in Theorem 3.10 shows that this happens if and only if $u(z) \to 0$ as $z \to \partial D$, completing our proof of the McDonald-Sundberg theorem. (This is not a circular proof of the McDonald-Sundberg theorem, as that result was used in the proof of Theorem 3.10 only in showing that (e) is equivalent to (a); this equivalence is not used in the proof we have just given).

The McDonald-Sundberg theorem proved in the paragraph above gives another example of how $\mathcal{U}$ provides a more natural context than $L^\infty(D, dA)$ for many Toeplitz operator questions. Specifically, the McDonald-Sundberg theorem just proved becomes false if the hypothesis that $u \in \mathcal{U}$ is weakened to the hypothesis that $u \in L^\infty(D, dA)$—Sarason constructed an example, presented in Section 5 of [12], of a function $u \in L^\infty(D, dA)$ such that T_u is compact but $|u(z)| = 1$ for all $z \in D$.

In Corollary 2.17, we showed that $S - T_{\tilde{S}}$ is in the commutator ideal $\mathcal{C}_\mathcal{T}$ for every $S \in \mathcal{T}$. This raises the question of when $S - T_{\tilde{S}}$ is a compact operator. In the corollary below, we answer this question for S lying in a dense subset of $\mathcal{T}$. We do not know whether the hypothesis on S in the corollary below could be replaced by the weaker hypothesis that $S \in \mathcal{T}$.

Corollary 3.12 *Suppose S is a finite sum of operators of the form $T_{u_1} \dots T_{u_n}$, where each $u_j \in \mathcal{U}$. Then the following are equivalent:*

(a) $S - T_{\tilde{S}}$ *is a compact operator;*

(b) $\tilde{S} \in$ *HOP;*

(c) $\lim_{z \to \partial D} (1 - |z|^2)^2 (\Delta \tilde{S})(z) = 0.$

Proof: In Theorem 2.2 of [6], we showed that a finite sum of finite products of Toeplitz operators is compact if and only if its Berezin transform has limit 0 on ∂D. Applying this result to the operator $S - T_{\tilde{S}}$, whose Berezin transform equals $\tilde{S} - \tilde{\tilde{S}}$, we conclude that $S - T_{\tilde{S}}$ is compact if and only if

$$\lim_{z \to \partial D} \tilde{S}(z) - \tilde{\tilde{S}}(z) = 0. \tag{3.13}$$

The equivalence of conditions (a) and (c) in Theorem 3.10 (with $u = \tilde{S}$) shows that (3.13) holds if and only if $\tilde{S} \in HOP$. In other words, conditions (a) and (b) above are equivalent.

Now suppose that (b) holds, so $\tilde{S} \in HOP$. Thus $\big(\Delta(\tilde{S} \circ \varphi_m)\big)(0) = 0$ for every $m \in \mathcal{M} \setminus D$. Proposition 2.26(c) (with $w = 0$) now tells us that $\lim_{z \to \partial D}(1 - |z|^2)^2(\Delta\tilde{S})(z) = 0$. In other words, (b) implies (c).

Now suppose that (c) holds. Fix $w \in D$ and $m \in \mathcal{M} \setminus D$. Note that $|\varphi_z(w)| \to 1$ as $z \to m$ (recall that w is fixed). From Proposition 2.26(c) and our hypothesis (c) we now conclude that $\big(\Delta(\tilde{S} \circ \varphi_m)\big)(w) = 0$. Thus $\tilde{S} \circ \varphi_m$ is harmonic on D. Because m was an arbitrary element of $\mathcal{M} \setminus D$, this means that $\tilde{S} \in HOP$. Thus (c) implies (b), completing the proof. ∎

4 Asymptotic Multiplicativity

The Berezin transform is not multiplicative even over the space of harmonic functions. However, $\widetilde{uv}(z) - \tilde{u}(z)\tilde{v}(z) \to 0$ as $z \to \partial D$ for some pairs of functions u, v. In this section we describe when this happens for bounded harmonic functions. Note that if u and v are harmonic, then $\tilde{u} = u$ and $\tilde{v} = v$, so we want to know when $\widetilde{uv}$ is approximately equal to uv near ∂D.

Our key tool in proving Theorem 4.5 will be Theorem 3.10. However we will also need the following two lemmas.

Lemma 4.1 *If u, v are bounded and harmonic on D, then*

$$\widetilde{uv}(z) - u(z)v(z) = (H_{\bar{u}}{}^* H_v + H_{\bar{v}}{}^* H_u)\tilde{}(z)$$

for every $z \in D$.

Proof: Suppose u and v are bounded and harmonic on D. There there are four functions $f_1, f_2, g_1, g_2 \in L^2_a(D)$ such that $u = f_1 + \bar{f}_2$ and $v = g_1 + \bar{g}_2$. Let $z \in D$. Then

$$\begin{aligned}
(H_{\bar{u}}{}^* H_v)\tilde{}(z) &= \langle H_{\bar{u}}{}^* H_v k_z, k_z\rangle \\
&= \langle H_v k_z, H_{\bar{u}} k_z\rangle \\
&= \langle (1-P)\big((g_1 + \overline{g_2})k_z\big), (1-P)\big((\overline{f_1} + f_2)k_z\big)\rangle \\
&= \langle (1-P)(\overline{g_2}k_z), (1-P)(\overline{f_1}k_z)\rangle \\
&= \langle \overline{g_2}k_z - \overline{g_2(z)}k_z, \overline{f_1}k_z - \overline{f_1(z)}k_z\rangle \\
&= \langle f_1\overline{g_2}k_z, k_z\rangle - f_1(z)\overline{g_2(z)}.
\end{aligned}$$

A similar formula holds for $(H_{\bar{v}}{}^* H_u)\tilde{}(z)$. Adding these two formulas gives

$$\begin{aligned}(H_{\bar{u}}{}^* H_v + H_{\bar{v}}{}^* H_u)\tilde{}(z) &= \langle (f_1 \overline{g_2} + \overline{f_2} g_1) k_z, k_z \rangle - f_1(z)\overline{g_2(z)} - \overline{f_2(z)} g_1(z) \\ &= \langle (f_1 + \overline{f_2})(g_1 + \overline{g_2}) k_z, k_z \rangle \\ &\quad - (f_1(z) + \overline{f_2(z)})(g_1(z) + \overline{g_2(z)}) \\ &= \langle uv k_z, k_z \rangle - u(z)v(z) \\ &= \widetilde{uv}(z) - u(z)v(z),\end{aligned}$$

as desired. ■

Although the following lemma is probably well known, we were unable to locate a proof in the literature. Thus we have included a proof.

Lemma 4.2 *Suppose u, v are harmonic on D. Then uv is harmonic on D if and only if at least one of the following conditions holds:*

(a) *u and v are both analytic on D;*

(b) *$\bar{u}$ and $\bar{v}$ are both analytic on D;*

(c) *there exist complex numbers α, β, not both 0, such that $\alpha u + \beta v$ and $\bar{\alpha}\bar{u} - \bar{\beta}\bar{v}$ are both analytic on D.*

PROOF: Because u and v are harmonic on D, an elementary computation shows that

$$\Delta(uv) = 4\left(\frac{\partial u}{\partial \bar{z}}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial \bar{z}}\right)$$

on D. Thus uv is harmonic if and only if

$$\frac{\partial u}{\partial \bar{z}}\frac{\partial v}{\partial z} = -\frac{\partial u}{\partial z}\frac{\partial v}{\partial \bar{z}}. \tag{4.3}$$

Clearly (a) implies that uv is harmonic, as does (b). Condition (c) can be restated to say that there exist complex numbers α, β, not both 0, such that

$$\alpha\frac{\partial u}{\partial \bar{z}} = -\beta\frac{\partial v}{\partial \bar{z}} \quad \text{and} \quad \alpha\frac{\partial u}{\partial z} = \beta\frac{\partial v}{\partial z}. \tag{4.4}$$

Thus (c) implies (4.3), proving one direction of the lemma.

To prove the other direction, we use an argument from the proof of Theorem 1 of [3]. Suppose that uv is harmonic, so (4.3) holds. Because u

and v are harmonic, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ are analytic; furthermore, $\frac{\partial u}{\partial \bar{z}}$ and $\frac{\partial v}{\partial \bar{z}}$ are conjugate analytic. Let

$$\Omega = \{w \in D : \frac{\partial v}{\partial z}(w) \neq 0 \text{ and } \frac{\partial v}{\partial \bar{z}}(w) \neq 0\}.$$

First consider the case where Ω is the empty set. Then either v is analytic or v is conjugate analytic. If v is analytic, then 4.3 implies $\frac{\partial u}{\partial \bar{z}}\frac{\partial v}{\partial z} = 0$, which implies that either u is analytic (so (a) holds) or v is conjugate analytic (so (c) holds with $\alpha = 0$ and $\beta = 1$). Similarly, if v is conjugate analytic, then either (b) or (c) holds, completing the proof when Ω is the empty set.

Now suppose Ω is not the empty set. Then Ω is a dense open subset of D. On Ω, we can rewrite 4.3 as

$$-\frac{\frac{\partial u}{\partial \bar{z}}}{\frac{\partial v}{\partial \bar{z}}} = \frac{\frac{\partial u}{\partial z}}{\frac{\partial v}{\partial z}}.$$

The left side of this equation is a conjugate analytic function on Ω. The right side is an analytic function on Ω. Thus both sides equal the same constant function on Ω. We conclude that for some constant β, we must have $\frac{\partial u}{\partial \bar{z}} = -\beta\frac{\partial v}{\partial \bar{z}}$ and $\frac{\partial u}{\partial z} = \beta\frac{\partial v}{\partial z}$. Thus (4.4) holds (with $\alpha = 1$) and hence (c) holds, completing the proof. ∎

Now we can describe when the Berezin transform is asymptotically multiplicative on harmonic functions. For partial results on when the Berezin transform is multiplicative on $\mathcal{B}(L^2_a)$, see Kiliç's paper [10].

Theorem 4.5 *Suppose u and v are bounded and harmonic on D. Then the following conditions are equivalent:*

(a) $\lim_{z \to \partial D} \widetilde{uv}(z) - u(z)v(z) = 0$;

(b) $\lim_{z \to \partial D} (1 - |z|^2)^2 \Delta(uv)(z) = 0$;

(c) *$uv \in HOP$;*

(d) *$2T_{uv} - T_uT_v - T_vT_u$ is compact;*

(e) *for each $m \in \mathcal{M} \setminus D$, at least one of the following conditions holds:*

- (i) *$u \circ \varphi_m$ and $v \circ \varphi_m$ are both in H^∞;*
- (ii) *$\bar{u} \circ \varphi_m$ and $\bar{v} \circ \varphi_m$ are both in H^∞;*
- (iii) *there exist complex numbers α, β, not both 0, such that $\alpha u \circ \varphi_m + \beta v \circ \varphi_m$ and $\bar{\alpha}\bar{u} \circ \varphi_m - \bar{\beta}\bar{v} \circ \varphi_m$ are both in H^∞.*

Proof: The equivalence of (a), (b), and (c) follows from the equivalence of conditions (a), (h), and (c) in Theorem 3.10.

The equivalence of (a) and (d) follows from Lemma 4.1 and Theorem 2.2 of [6] along with the identity $2T_{uv} - T_u T_v - T_v T_u = H_{\bar{u}}{}^* H_v + H_{\bar{v}}{}^* H_u$.

Finally, the equivalence of (c) and (e) follows from Lemma 4.2. ∎

As an application of the theorem above, we now show how it can be used to give an easy proof of the characterization of the functions $f, g \in H^\infty$ such that $T_{\bar{f}} T_g - T_g T_{\bar{f}}$ is compact. Suppose $f, g \in H^\infty$. Then

$$2T_{\bar{f}g} - T_{\bar{f}} T_g - T_g T_{\bar{f}} = T_{\bar{f}} T_g - T_g T_{\bar{f}}.$$

Thus by Theorem 4.5 (with $u = \bar{f}$ and $v = g$), $T_{\bar{f}} T_g - T_g T_{\bar{f}}$ is compact if and only if

$$\lim_{z \to \partial D} (1 - |z|^2)^2 \Delta(\bar{f} g)(z) = 0.$$

Because $\Delta = 4(\partial/\partial z)(\partial/\partial \bar{z})$, we see that $T_{\bar{f}} T_g - T_g T_{\bar{f}}$ is compact if and only if

$$\lim_{z \to \partial D} (1 - |z|^2)^2 f'(z) g'(z) = 0. \tag{4.6}$$

This result was originally proved by Zheng [14] using other methods; also see [4] for additional conditions that are equivalent to (4.6).

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